Mathematics Club, IITM
Presents

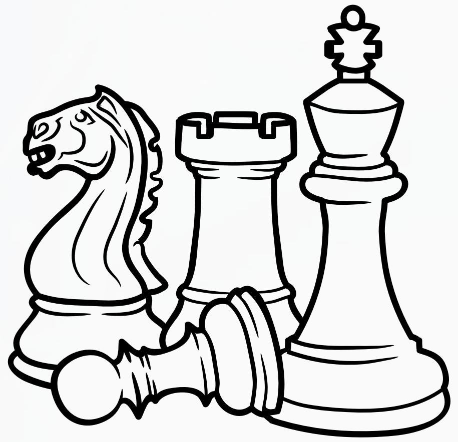
Strategy, Games and Rivalry

S.G.R.

Sequential vs Simultaneous

In sequential games , players observe what rivals have done in the past and there is a specific order of play.

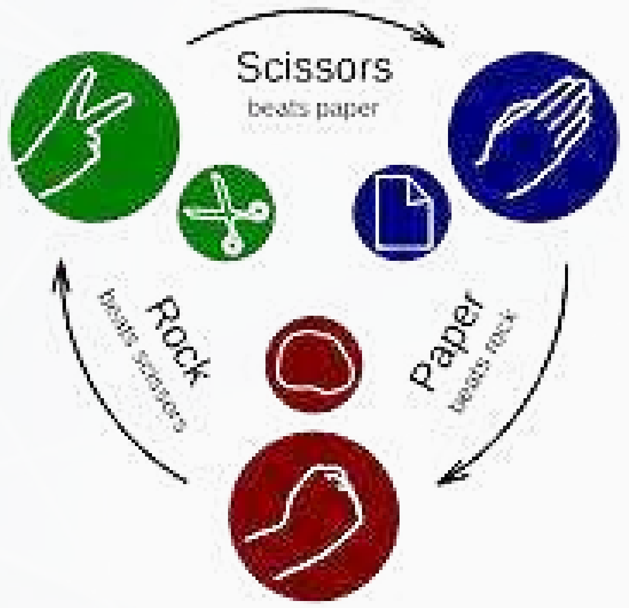
Example : Chess



Sequential vs Simultaneous

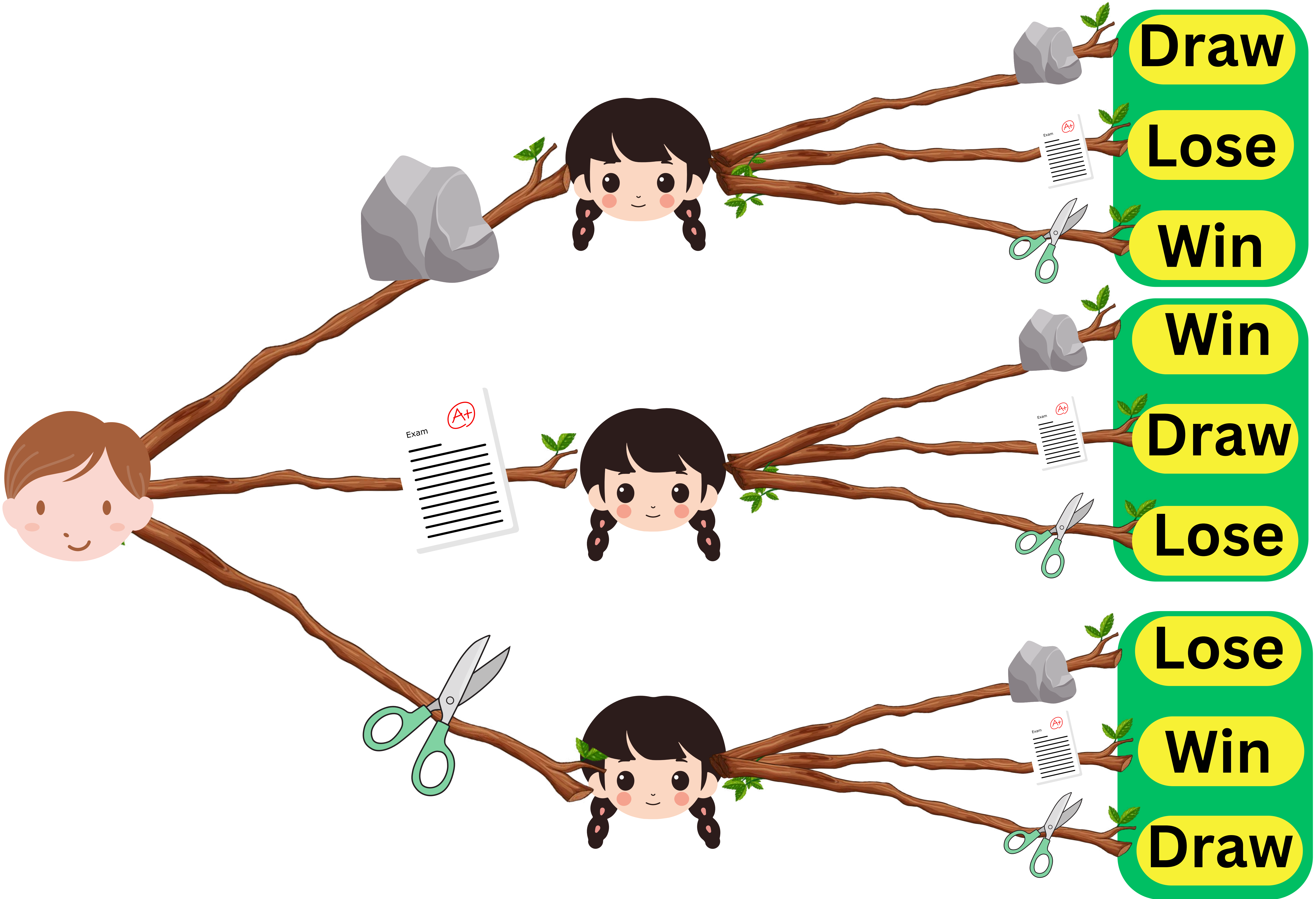
In simultaneous games , all players select strategies without observing the choices of their rivals and players choose at the exact same time.

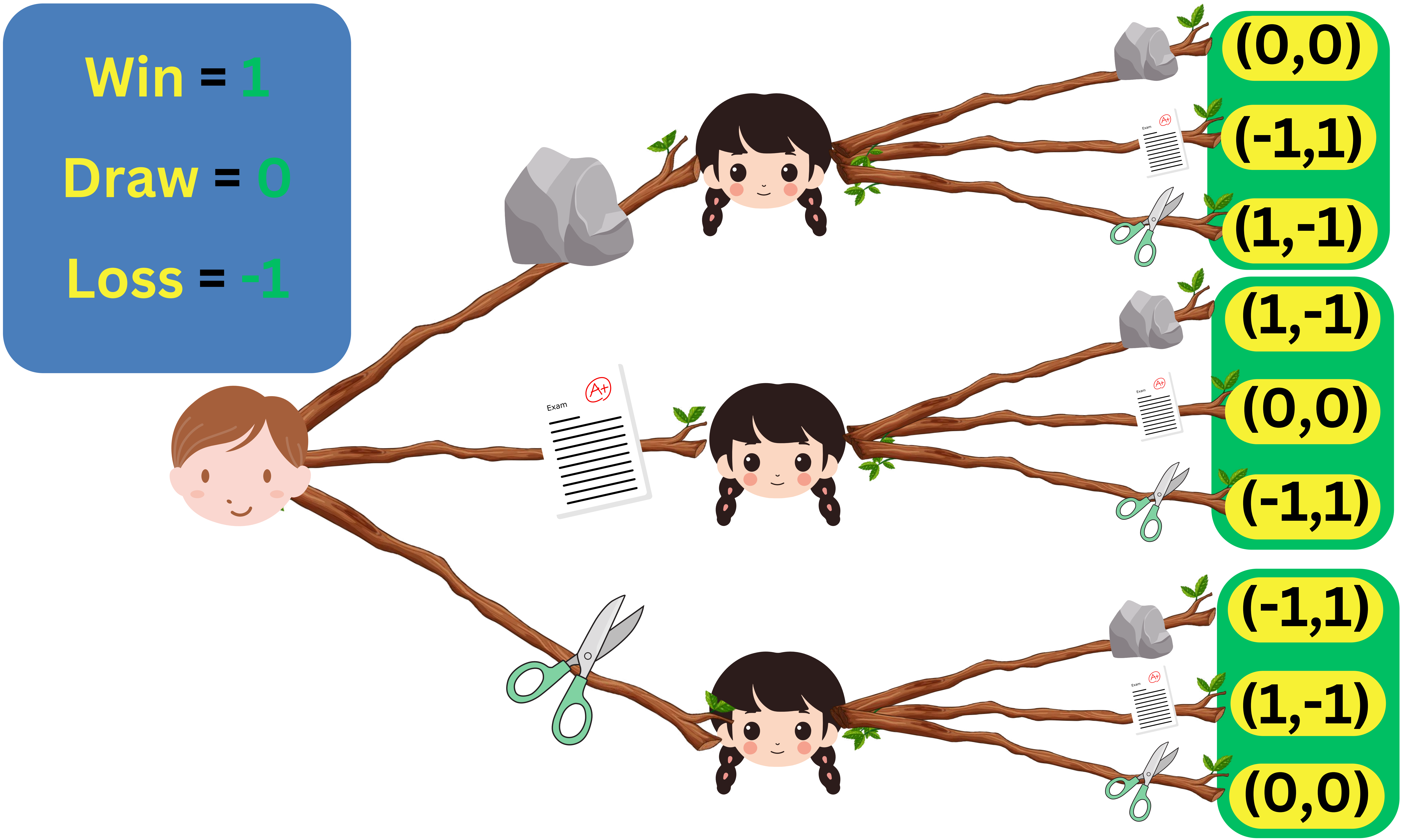
Example : Rock Paper Scissors



Game Tree

Let us create a Rock-Paper-Scissors Game Tree !


Exam
A+
Draw
Lose
Win
Win
Draw
Lose
Lose
Win
Draw


Win = 1
Draw = 0
Loss = -1
Exam
A+
(0,0)
(-1,1)
(1,-1)
(1,-1)
(0,0)
(-1,1)
(-1,1)
(1,-1)
(0,0)

Let's Play

Players

Harry

Hermione

Ron

Moves

Harry : Run or Not ?

Hermione : Jump or Not ?

Ron : Shoot or Not ?

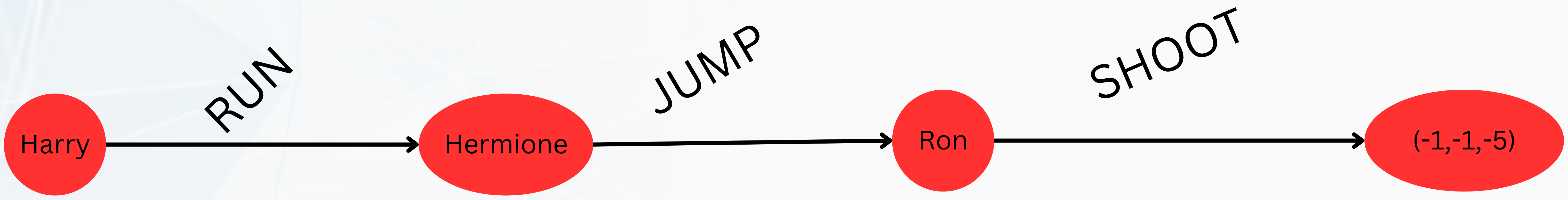
Game - Type

Sequential

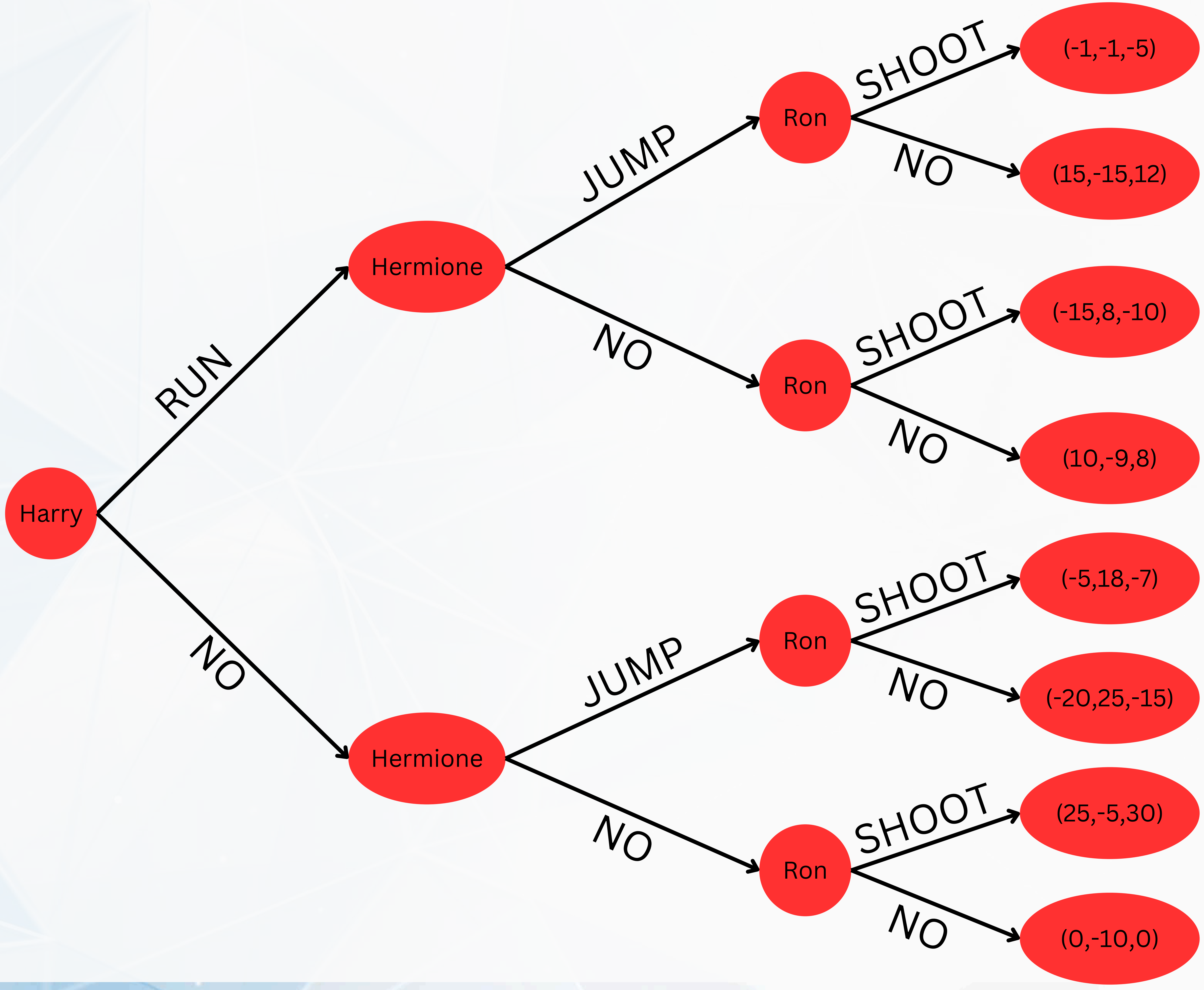
Harry

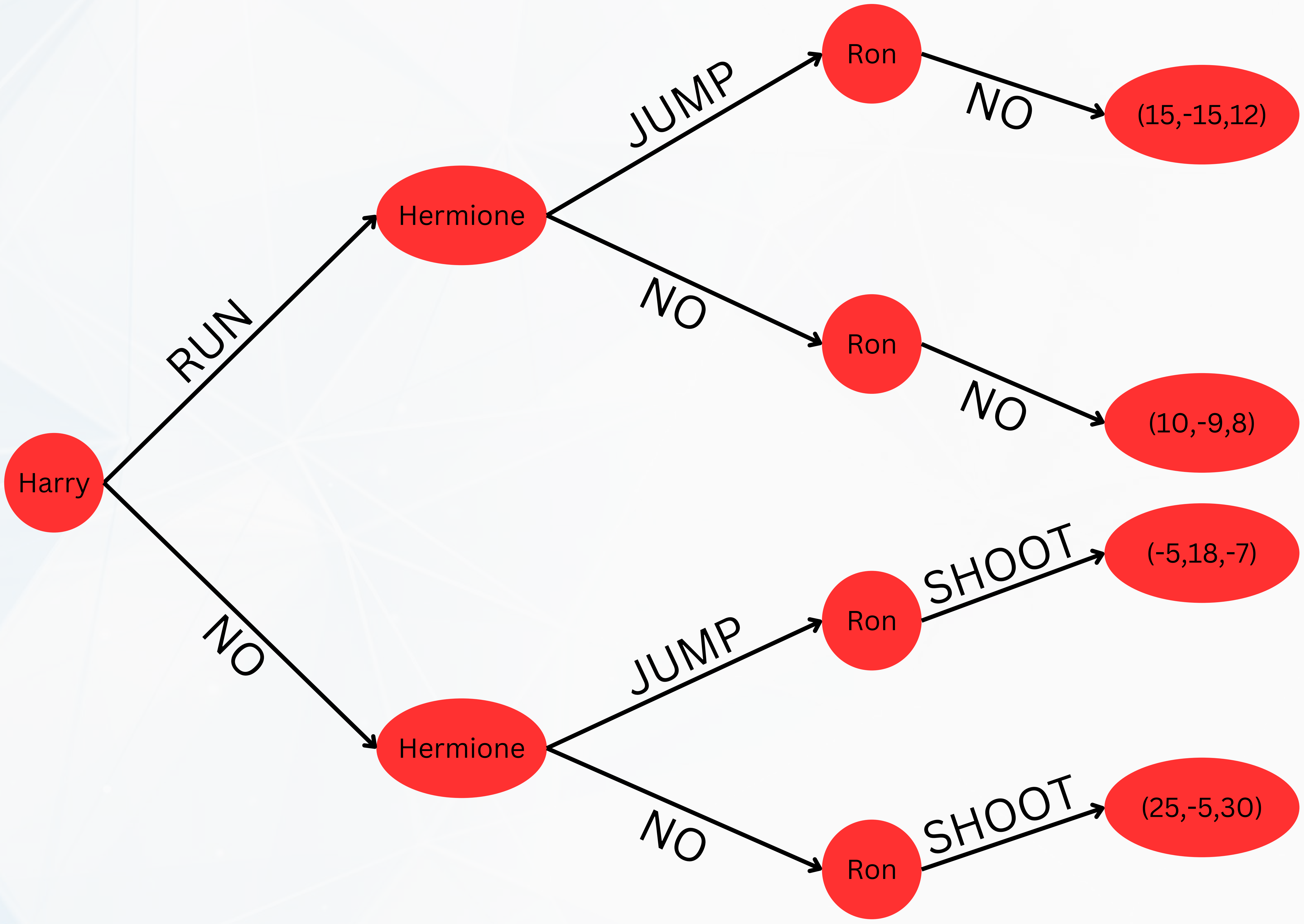
Hermione

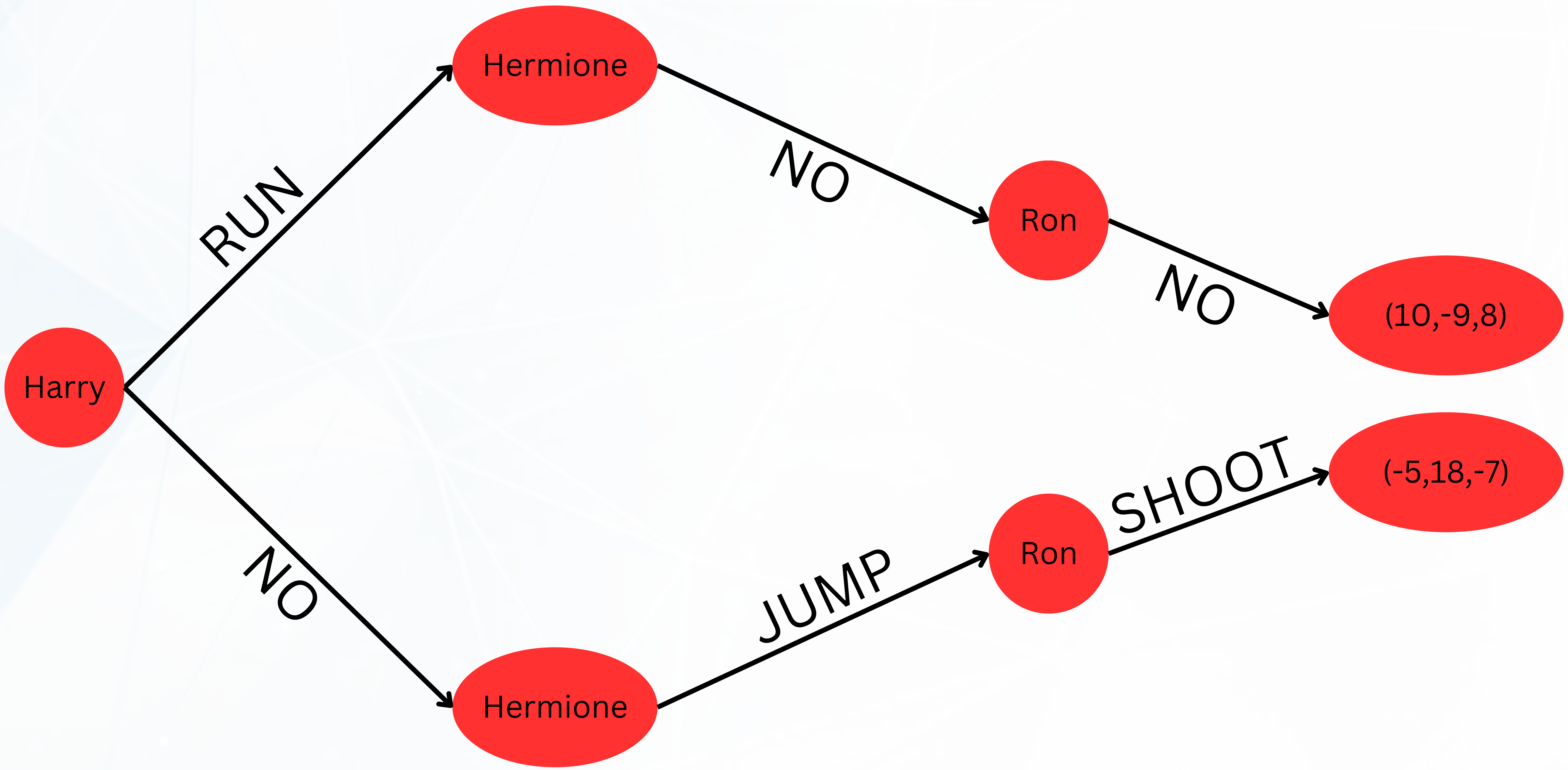
Ron

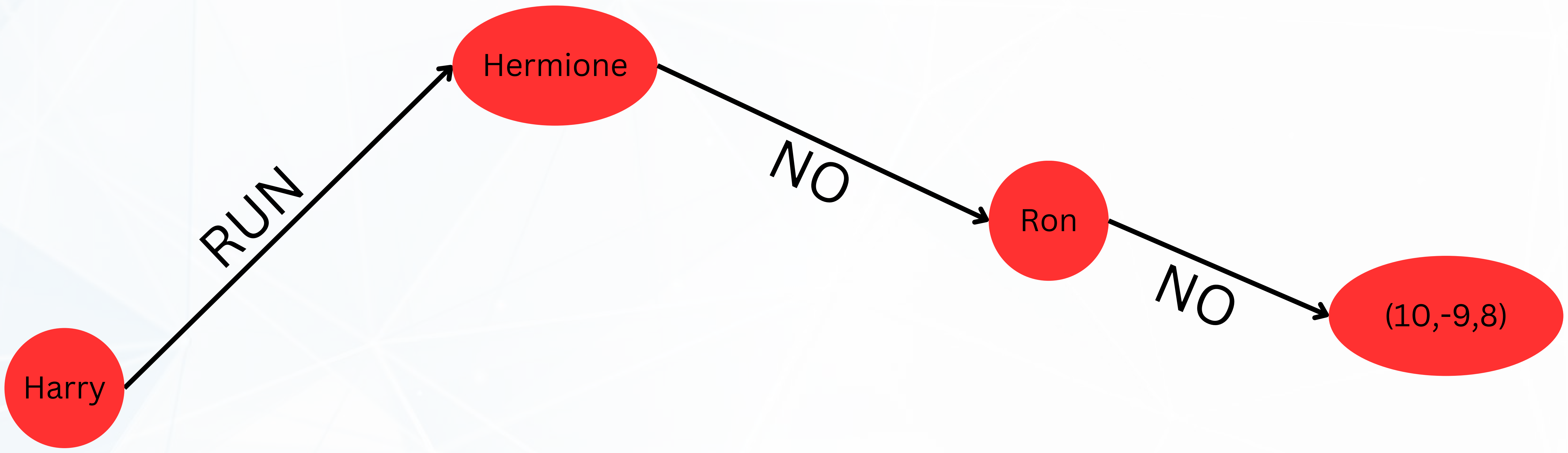


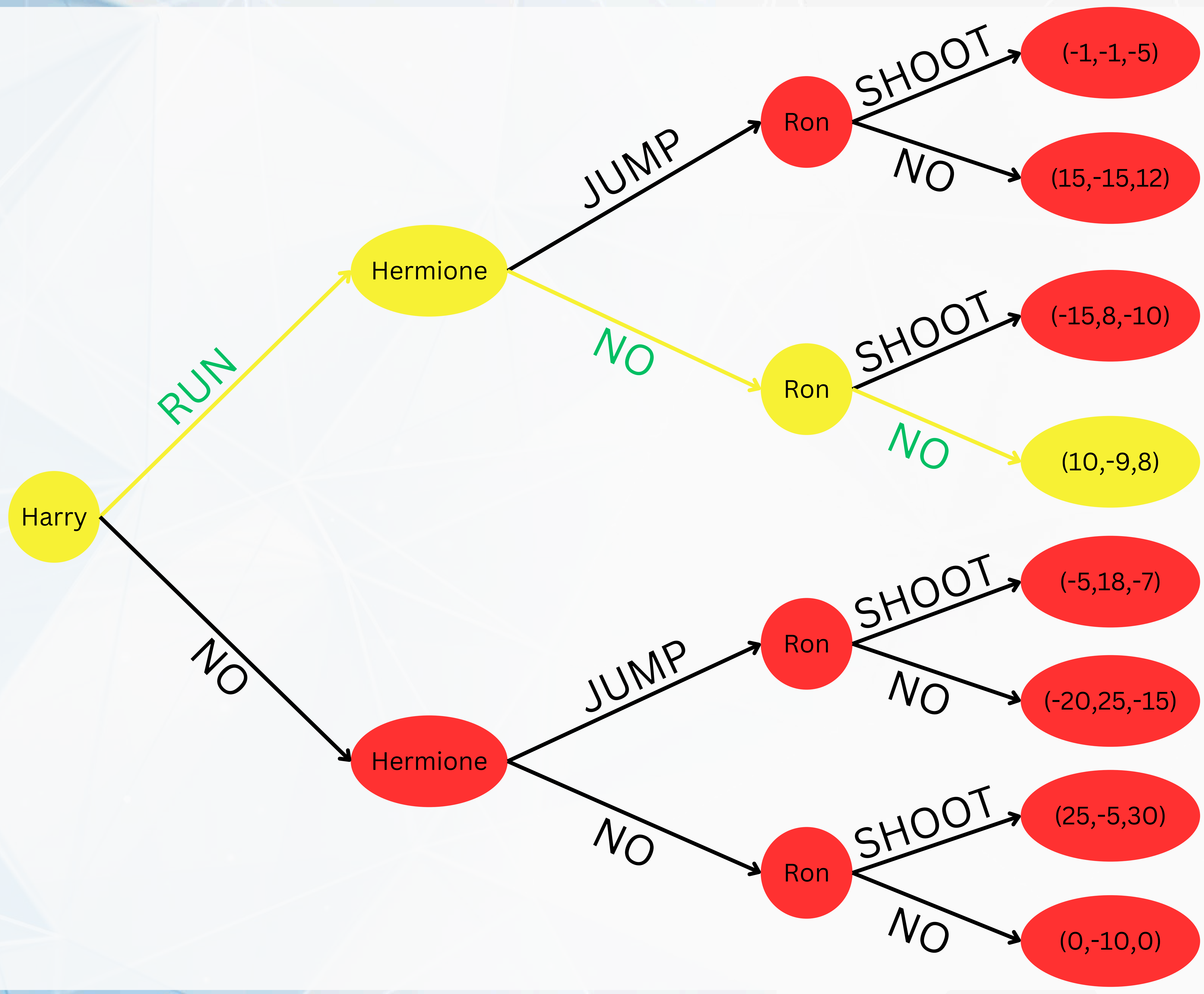
Harry
RUN
Hermione
JUMP
Ron
SHOOT
(-1,-1,-5)


Harry
RUN
NO
Hermione
Hermione
JUMP
NO
JUMP
NO
Ron
Ron
Ron
Ron
SHOOT
NO
SHOOT
NO
SHOOT
NO
SHOOT
NO
(-1,-1,-5)
(15,-15,12)
(-15,8,-10)
(10,-9,8)
(-5,18,-7)
(-20,25,-15)
(25,-5,30)
(0,-10,0)


Harry
RUN
NO
Hermione
Hermione
JUMP
NO
JUMP
NO
Ron
Ron
Ron
Ron
NO
NO
SHOOT
SHOOT
(15,-15,12)
(10,-9,8)
(-5,18,-7)
(25,-5,30)


Harry
RUN
NO
Hermione
Hermione
NO
JUMP
Ron
Ron
NO
SHOOT
(10,-9,8)
(-5,18,-7)


Harry
RUN
Hermione
NO
Ron
NO
(10,-9,8)


Harry
RUN
NO
Hermione
JUMP
NO
Hermione
JUMP
NO
Ron
SHOOT
NO
Ron
SHOOT
NO
Ron
SHOOT
NO
Ron
SHOOT
NO
(-1,-1,-5)
(15,-15,12)
(-15,8,-10)
(10,-9,8)
(-5,18,-7)
(-20,25,-15)
(25,-5,30)
(0,-10,0)

Game Tree - Backwards Induction

Illusion of Control

Harry may be tempted to choose (25,-5,30) but Should he?
Can he?

First is Last ?

Ron makes the **FINAL** choice !

Start From End

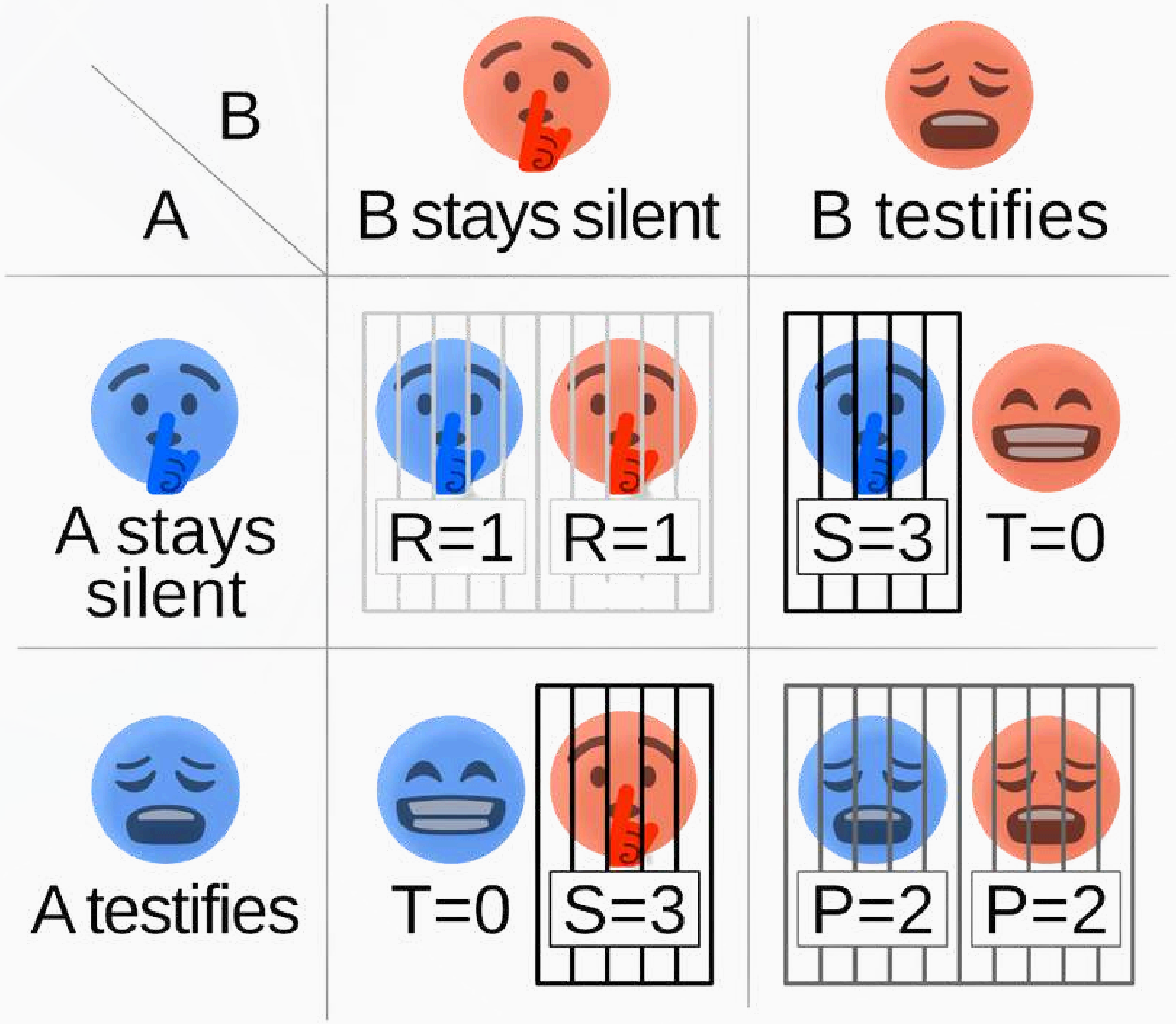
Harry can should make his decision based on Ron's and Hermione

Prisoner's Dilemma

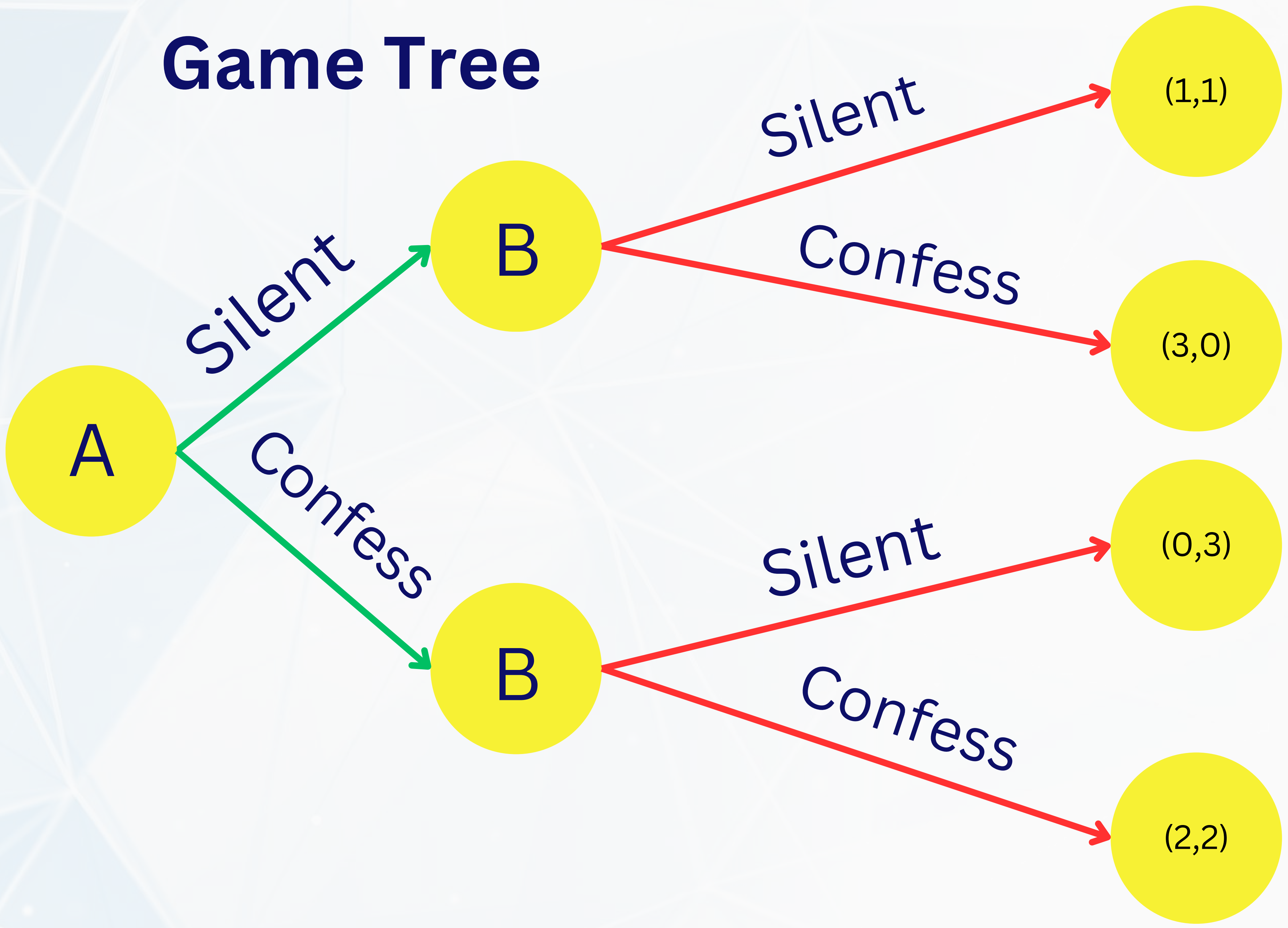
(A Simultaneous game)

A popular hypothetical scenario featuring two prisoners

Both players get out early if they stay silent (But should they?)

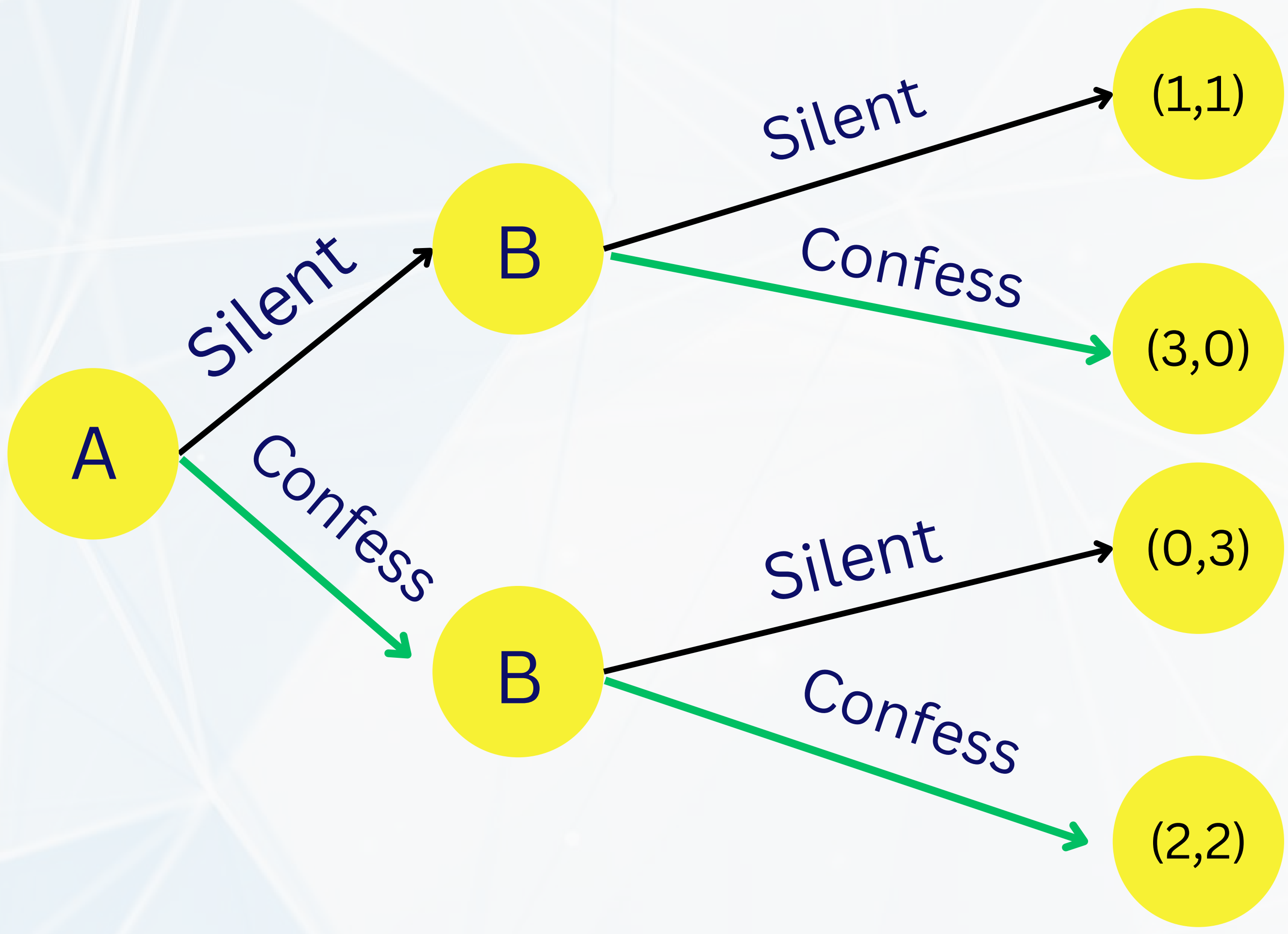


Game Tree



Green Edges: A’s choices
Red Edges: B’s choices

Solve Prisoner's Dilemma using the Game Tree



Let us consider B's choice first. B plays his turn independent of A's choice. B's only goal is to minimise his own sentence. We can see that irrespective of A's choice, the optimal move for B is to confess (as it minimises his sentence in both cases)

The same is true for A

Appreciating Prisoner's Dilemma

Thus the optimal choice for both player's is to betray each other which results in both of them losing!

This concept is used in a real life game show

Split or Steal:

A prize is divided between two players, putting their friendship to the test

Result	Split		Steal	
Split	50%	50%	100%	0%
Steal	0%	100%	0%	0%

Ever played 21 dares?

Rules:

- Players start counting from 1
- On their turn, a player can say a minimum of 1 and a maximum of 3 numbers
- The player to reach 21 loses

21 dares with two players

Tired of doing dares. You friend might have been cooking some strategies

21 Dares - Winning and losing States

Winning state ⟹ You are guaranteed to win the game or move to another winning state (if you play optimally) irrespective of your opponents choice.

Losing state ⟹ You will lose or will move to another losing state (if your OPPONENT plays optimally) irrespective of your choice.

Do what you gotta do.

Win = Winning State

How to Win ?

Reach a winning state !!!
(Then you can move from one winning state from another!)

20 is a winning state

If we reach 20 our opponent has to take 21 and he/she loses.

Can there be more than one winning state?

Do what you gotta do.

Win = Winning State

What is the previous winning state ?

We need to reach 16 if we need to reach 20 . Because 16 leads to 20, 16 is also a winning state.

OTHER WINNING STATES

$$16 \Rightarrow 12 \Rightarrow 8 \Rightarrow 4 \Rightarrow 0$$

Question ?

But what does 0 mean?

Answer :

Let your opponent start the game

😉

Did you realise ?

Reverse Thinking

This process of solving from the <u>**FINAL**</u> state is called

Backward Induction.

Divide and Conquer

We divided the game into smaller games that are called

Sub-Games.

Nim game

Initial condition

Given a number of piles in which each pile contains some numbers of stones/ coins.

Moves

In each turn, a player can choose only one pile and remove any number of stones (at least one) from that pile.

Objective

The player who takes the last coin wins the game!!

Keep calm and Xor on.

XOR operation

Input A	Input B	Output
0	0	0
0	1	1
1	0	1
1	1	0

XOR on decimal numbers

- Just take each bit of the number and perform XOR

$$
5 \oplus 1 = \begin{array}{ccc} 1 & 0 & 1 \\ \oplus & \oplus & \oplus \\ 0 & 0 & 1 \\ \\ 1 & 0 & 0 \end{array} = 4
$$

XOR Sum

01 Let us define XOR sum of numbers as

$$\text{XOR sum}(a_1, a_2, \ldots a_n)$$

$$= a_1 \oplus a_2 \oplus a_3 \oplus \ldots a_n$$

$$\text{Example} : 4 \oplus 5 \oplus 2 \text{ is}$$

$$\begin{array}{rl} & 1\,0\,0 \;\oplus \\ & 1\,0\,1 \;\oplus \\ & 0\,1\,0 \\ = & 0\,1\,1 \end{array}$$

But how?

01 When will XOR sum of a set of numbers become 0?

Let us start of with 2 bit numbers

$$1 \oplus 3 \oplus 2 = 0$$

$$\begin{array}{rl} & 0\,1 \quad \oplus \\ & 1\,1 \quad \oplus \\ & 1\,0 \\ = & 0\,0 \end{array}$$

But how?

01 When will XOR sum of a set of numbers become 0?

Example with 3 bits

$$4 \oplus 3 \oplus 5 \oplus 2 = 0$$

$$\begin{array}{rl} & 1\,0\,0 \oplus \\ & 0\,1\,1 \oplus \\ & 1\,0\,1 \oplus \\ & 0\,1\,0 \\ = & 0\,0\,0 \end{array}$$

Go optimal or go home.

XOR sum being zero

- If the XOR sum of 'n' numbers will be zero if and only if the bit **1** appears even number of times at each bit position.

$$
\begin{array}{rl}
 & 0\,1 \oplus \\
 & 1\,1 \oplus \\
 & 1\,0 \\
= & 0\,0
\end{array}
$$

$$
\begin{array}{rl}
 & 1\,0\,0 \oplus \\
 & 0\,1\,1 \oplus \\
 & 1\,0\,1 \oplus \\
 & 0\,1\,0 \\
= & 0\,0\,0
\end{array}
$$

XOR Sum

02

How is XOR sum related to Nim game?
Is the solution dependent on XOR sum?

Hint: Take XOR sum of number of coins in each pile

XOR Sum

What is the XOR sum of the number of coins in each pile at winning state?

The winning state for this game is achieving 0 as number of coins in each pile

$$0 \oplus 0 \oplus 0 \cdots \oplus 0 = 0$$

XOR Sum in nim game

What happens to parity of number of **1** bit at each bit position after an operation in nim game.

The parity of the bit in atleast one position changes after each move!!

Example : Let us take 2 piles with 3 and 2 coins

1 1

1 0

Go optimal or go home.

XOR sum being zero

- If the XOR sum of ‘n’ numbers is already zero then there is no possibility to make the XOR sum zero by single reduction of a number.

CAN YOU GUESS WHY NOT??

XOR sum being zero

- In this case, for any number of coins in any pile that player 1 chooses, player 2 can choose a suitable number of coins from another pile such that the xor remains to be 0. Hence if played optimally, Player 2 always wins.

XOR sum being non-zero

- If the XOR sum of ‘n’ numbers is non-zero then there is at least a single approach by which if you reduce a number, the XOR sum is zero (can you prove this?)

Hint: Try to find the number!!

Go optimal or go home.

XOR sum being non-zero

- In this case, Player 1 can choose a suitable number of coins from a suitable pile to get the xor to be 0. This is equivalent to the previous case but with Player 2 starting, hence as we have seen, Player 1 always wins in this case if played optimally.

We are the Mathematics Club

WEBSITE

WHATSAPP

INSTAGRAM

Thank You

By ∞ Team