BeReal
Solution Key

Mathematics Club

11 March 2025

§1 Are you even Continuous Bro? (2 marks)

Prove that there exists no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) \in \mathbb{Q} \iff f(x+1) \in \mathbb{R} \backslash \mathbb{Q}$$

Solution:

Consider the continuous function $g(x) = f(x+1) - f(x)$. By our hypothesis it follows that g has only irrational values, that is, g is constant. Let $a \in \mathbb{R} \setminus \mathbb{Q}$ be such that $g(x) = a$, for all $x \in \mathbb{R}$. This means that

$$f(x+2) - f(x) = 2a \in \mathbb{R} \setminus \mathbb{Q}, \quad \forall x \in \mathbb{R}.$$

On the other hand, let $x_0 \in \mathbb{R}$ be such that $f(x_0) \in \mathbb{Q}$. Considering a continuous function h given by $h(x) = f(x+1) + f(x)$, and noticing that h is constant, we obtain $f(x_0+2) \in \mathbb{Q}$ (by equating $h(x)$ to $h(x+1)$). So $f(x_0+2) - f(x_0) \in \mathbb{Q}$. But we have just shown above that $f(x+2) - f(x) = 2a \in \mathbb{R} \setminus \mathbb{Q}, \quad \forall x \in \mathbb{R}$. This contradiction shows that such a function cannot exist. **(2 marks for full sol, no partials)** □

§2 You've been struck by... (1+1 mark)

Comment on the Differentiability and Analyticity of the function f.

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

Solution:

We already know the function is differentiable everywhere except at origin.

(1 mark for showing Continuity and Differentiability at origin)

The function f is **smooth**, and all its derivatives at the **origin** are 0. Therefore, the **Taylor series** of f at the origin converges everywhere to the **zero function**,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0, \quad x \in \mathbb{R},$$

and so the Taylor series does not equal $f(x)$ for $x > 0$. Consequently, f is **not analytic** at the origin. **(1 mark)**

§3 Atrocious Assumption (2 marks)

Ashwin is excited as he just "found out" that the sequence $a_n = \sin(n)\ \forall\ n \geq 1$ converges to the value $\cos(1/2)$ by simply solving $a_n = a_{n+1}$. He won't accept that he is wrong unless you give him a proof of divergence of the sequence.

Prove that the sequence $(a_n)_{n\geq 1}$ is divergent.

Solution:

Arguing by contradiction, we assume that the sequence $(a_n)_{n\geq 1}$ is convergent. Let $a = \lim_{n\to\infty} \sin n$. Using the identity

$$\sin(n+1) = \sin n \cos 1 + \cos n \sin 1$$

we deduce that the sequence $(\cos n)$ converges and, moreover,

$$a = a\cos 1 + b\sin 1,$$

where $b = \lim_{n\to\infty} \cos n$. Using now

$$\cos(n+1) = \cos n \cos 1 - \sin n \sin 1$$

we deduce that

$$b = b\cos 1 - a\sin 1.$$

These two relations imply $a = b = 0$, a contradiction, since $a^2 + b^2 = 1$. □

(2 marks for full solution, no partials)

§4 Convergence Curse (2 marks)

Swaminath has pulled an all-nighter studying the behaviour of the sequence of functions $f_n(x) = \sum_{k=0}^{n} x^k$.

He was told by Aayush that f_n does not converge uniformly in the interval $x \in [0,1)$ but converges locally uniformly in the interval $x \in [0,1)$. He comes to you for help as he is still stumped! Show that f_n follows the properties mentioned above.

Solution:

Uniform Convergence (informal definition):

For uniform convergence we must be able to find $f(x)$ such that such that $||f_n(x) - f(x)|| < \epsilon$ for all $n \geq$ some n_0 at all points in the domain $[0,1]$

In the interval $[0,\lambda]\ \forall\ \lambda < 1$ The sequence f_n converges to $f(x) = \frac{1}{1-x}$. If the sequence converges uniformly then $f(x)$ must be continuous (since uniform convergence carries forward analyticity).
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{1-x}$ which is infinite, and $f(x)$ is discontinuous there.
Thus there exists no such n_0, i.e. the sequence is **Not Uniformly Convergent (1 mark)**

Proving Local convergence:
For intervals $[0,\lambda]\ \forall\ \lambda < 1$ one has to show that the sequence converges to $f(x) = \frac{1}{1-x}$ using $\epsilon - \delta$ method over every compact closed subset.

If they show that for any compact closed subset $[0,\beta]$ or $[\alpha,\beta]$ where $\alpha > 0$ and $\beta < 1$ the sequence converge, then they have shown that the sequence is **Locally Uniformly Convergent (1 mark)**

§5 Back to School Algebra (2 marks)

Smitali has now given you all a simple challenge: find one of the roots of the equation

$$x^n + x + a = 0 \quad \text{where } n \geq 1 \text{ is an odd integer}$$

Recall Pranjal teaching the Taylor's formula for series expansion of a given infinitely differentiable function. Also recall that the Taylor expansion coincides with the function when the function in question is analytic. You now go to Pakshal for help, who introduces you to the "Lagrange Inversion Formula" for the series expansion of the inverse of a function:

Suppose z is defined as a function of w by an equation of the form: $z = f(w)$, where f is analytic at a point a and $f'(a) \neq 0$. Then it is possible to solve the equation for w, expressing it in the form $w = g(z)$, where $g = f^{-1}$ given by a power series (if it exists in a suitable domain):

$$g(z) = a + \sum_{n=1}^{\infty} g_n \frac{(z - f(a))^n}{n!} \quad \text{where} \quad g_n = \lim_{w \to a} \frac{d^{n-1}}{dw^{n-1}} \left[\left(\frac{w-a}{f(w) - f(a)} \right)^n \right]$$

Use the Lagrange Inversion formula to find one of the the roots of Smitali's equation. Assume that the root lies in the interval of convergence of the series. (It is sufficient to represent the root as a summation, without involving any derivative operators or limits in the expression)

Solution:

Consider $f : \mathbb{R} \to \mathbb{R}$ given by, $f(x) = x^n + x$. This is always invertible for odd n. We want to find $f^{-1}(-a)$ using Lagrange inversion.

Expand about 0, to get

$$g_k = \lim_{x \to 0} \frac{d^{k-1}}{dx^{k-1}} \left(\frac{x}{x^n + x} \right)^k = \lim_{x \to 0} \frac{d^{k-1}}{dx^{k-1}} \left(1 + x^{n-1}\right)^{-k} = \lim_{x \to 0} \frac{d^{k-1}}{dx^{k-1}} \sum_{r=0}^{\infty} \binom{k+r}{r} (-1)^r x^{(n-1)r}$$

For a non-zero g_k, we need $r(n-1) = k - 1$. So, $g_k = g_{r(n-1)+1} = (-1)^r \binom{r(n-1)+1+r}{r} (r(n-1))!$ For all other k, the coefficient is simply 0. Thus,

$$g(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(rn+1)!}{r!} \frac{x^{r(n-1)+1}}{(r(n-1)+1)!} = \sum_{r=0}^{\infty} \binom{rn+1}{r} \frac{(-1)^r}{r(n-1)+1} x^{r(n-1)+1}$$

The root is

$$\sum_{r=0}^{\infty} \binom{rn+1}{r} \frac{(-1)^r}{r(n-1)+1} (-a)^{r(n-1)+1}$$

(2 marks for full solution, no partials)

§6 Deja Vu (3 marks)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ continuous at 0 satisfying the following relation

$$f(x+y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$$

Solution:

Clearly $f(0) = 0$ by induction we have for all integers n $f(nx) = nf(x)$. For some rational number $x = \frac{p}{q}$ we have $f(qx) = f(p) = pf(1)$ and also $f(qx) = qf(x)$ which together gives $f(x) = xf(1)$ for all rational numbers. **(1 mark)**

Now since function is continuous at 0. For all $\epsilon > 0$ we have $\delta > 0$ such that

$$|h - 0| < \delta \implies |f(h + 0) - f(0)| = f(h) < \epsilon$$

which means $f(h) \to 0$ as $h \to 0$

From the above limiting behavior of $f(x)$ around 0 we find that for all $\epsilon > 0$ and $x \in \mathbb{R}$ there exists $\delta > 0$ such that

$$|h| < \delta \implies |f(h + x) - f(x)| = f(h) < \epsilon$$

which means f is continuous on the whole real line. **(1 mark to prove f is continuous everywhere)**

For any irrational number x there exists a converging sequence of rational numbers $\{a_i\}_{i\in\mathbb{N}}$ such that $\lim a_n = x$. From limit criterion of convergence we have

$$\lim_{t\to x} f(t) = \lim_{i\to\infty} f(a_i) = \lim_{i\to\infty} a_i f(1) = af(1)$$

(1 mark for showing that irrationals follow the trend too. This is invalid if continuity isn't shown (no marks in that case))

Thus all solutions are of the form $f(x) = kx$ **(Writing this explicitly isn't compulsory for marks)**

A common mistake that we encountered was the use of L'Hopital's rule. This would be accepted only if you showed that the function was also differentiable everywhere. Assuming beforehand that the function is differentiable would lead to zero marks.

§7 Shrinking or Growing? (3 marks)

Ananya defines a sequence $\{x_n\}_{n=1}^{\infty}$ by the following recurrence relation:

$$x_{n+1} = \frac{1}{5}e^{-2x_n} + \frac{4}{5}x_n$$

He conjectures that the sequence converges for $x_1 > 0$. Can you prove or disprove his claim?

Solution:

Consider the function $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{5}e^{-2x} + \frac{4}{5}x$

Then, its derivative $f'(x) = \frac{-2}{5}e^{-2x} + \frac{4}{5}$. It follows that $|f'(x)| \leq \frac{4}{5}$ $\forall x > 0$ **(1 mark for showing derivative is bounded by a real number less than 1)**

Now,

$$\frac{f(x) - f(y)}{x - y} = f'(c) \text{ for some } c \in (x, y) \text{ by Lagrange's MVT} \implies \frac{|f(x) - f(y)|}{|x - y|} \leq \frac{4}{5} \quad \forall x, y > 0$$

(1 mark for LMVT or other ways to show contraction map)

Thus f is a contraction map. By Banach, the sequence $\{x_1, f(x_1), f(f(x_1)), ..., f^n(x_1), ...\}$ must converge to a unique fixed point. Thus (x_n) given by $x_{n+1} = f(x_n)$ converges.
(1 mark for final ansswer using Banach's or some alternate method)

(Full 3 marks if an entirely different method is used) ~~In this case give paper to Navin for checking~~

Alternative solution:

Some answers have "assumed" beforehand that the sequence converges to an x^* where $x^* = e^{-2x^*}$. This is ok, as long as you show with proper justification that no matter what x_1 you begin the sequence with, the sequence will always converge to x^*. The existence of an answer to the equation $x_{n+1} = x_n$ does not mean that the sequence (x_n) converges. Failure to realise this would lead to zero marks.

§8 Means are Tiny (1 + 1 mark)

For a sequence (a_n), Jash defines $S_n = \frac{1}{n}\sum_{k=1}^{n} a_k$.

a) Can you help him show that:

$$\limsup_{n\to\infty} S_n \leq \limsup_{n\to\infty} a_n$$

b) Why do you think that the lim sup is needed here? Why can't we use lim instead? Explain by giving an example where the two are not the same.

Solution:

By definition of the lim sup, there exists an N such that for all $k > N$, we have $a_k \leq \limsup a_n$. Now

$$S_n = \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{N} a_k + \frac{1}{n}\sum_{k=N+1}^{n} a_k \leq \frac{1}{n}\sum_{k=1}^{N} a_k + \frac{1}{n}\sum_{k=N+1}^{n} \limsup a_n$$

$$S_n \leq \frac{1}{n}\sum_{k=1}^{N} a_k + \frac{n-N}{n}\limsup a_n \implies \limsup_{n\to\infty} S_n \leq 0 + 1 \times \limsup_{n\to\infty} a_n \quad \text{(since N is finite)}$$

(1 mark)

A **common misunderstanding** that we encountered was the claim that taking the limit as $n \to \infty$ on both sides of the equation $x_n \leq \sup(y_n)$ would result in $\lim_{n\to\infty}(x_n) \leq \limsup_{n\to\infty}(y_n)$. It is very important to realize that $\limsup(\textit{something}) \neq \lim(\sup(\textit{something}))$.

Note that what the limit supremum of a sequence, that is $\limsup_{n\to\infty}(x_n)$ means by standard convention is $\lim_{n\to\infty} \sup(x_m)_{m\geq n}$. Consider the sequence (x_n) given by $x_n = 1 \quad \forall n \geq 100$ and $x_n = n \quad \forall 0 < n < 100$. Now we have $\sup(x_n) = 100$. Taking the limit as $n \to \infty$ of this constant 100 is trivial and would yield 100 itself, while $\limsup_{n\to\infty}(x_n) = 1$ because after the initial perturbations the sequence only contains a collection of 1s, whose supremum is 1.

A simple example where lim sup is relevant is the sequence $a_n = 1$ for even n, $a_n = 0$ for odd n. The lim is not even defined but the lim sup is well defined. **(1 mark)**

§9 BeComplex? (Bonus: 3 marks)

Recall Shivanshu teaching you that for a derivative operator to be defined for a given class of functions, the operator must be linear. If $f, g : \mathbb{R}^n \to \mathbb{R}^m$, and D is the derivative operator defined for these functions then,

$$D\{\alpha f + \beta g\} = \alpha D\{f\} + \beta D\{g\} \quad \forall \text{ constant } \alpha, \beta$$

As such, a derivative can be defined for functions mapping $\mathbb{R}^2$ to $\mathbb{R}^2$. Keeping in mind that the complex numbers $\mathbb{C}$ can be represented using $\mathbb{R}^2$, Anirudh comes up with this interesting question.

Are the class of functions $\{f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is differentiable$\}$ and the class of functions $\{g : \mathbb{C} \to \mathbb{C}$ such that g is differentiable$\}$ the same? Help answer Anirudh's question.

Solution:

No, they are not the same. Let $f(x, y) = (u, v)$ and $g(x + iy) = u + iv$.

The additivity $D\{f + g\} = D\{f\} + D\{g\}$ holds true for both. The crucial difference between the derivatives for f and g lies in the fact that D must be homogeneous in constants α, β from $\mathbb{R}$ for f while homogeneous in constants from $\mathbb{C}$ for g. **(1 mark for identifying α, β are complex)**

Let us assume that D is the $\mathbb{R}^2$-derivative which has been extended to $\mathbb{C}$. We want this derivative to have in addition to its own properties, $D\{\alpha g\} = \alpha D\{g\} \forall \alpha \in \mathbb{C}$.

A formal proof of the necessary and sufficient condition for an $\mathbb{R}^2$-derivative to be a $\mathbb{C}$-derivative is given here. This proof may be omitted if necessary. Expand $\alpha = a + ib$ and $g = u + iv$ and substitute to get,

$$D\{\alpha g\} = D\{au - bv + i(bu + av)\} = aD\{u\} - bD\{v\} + bD\{iu\} + aD\{iv\}$$

$$\alpha D\{g\} = (a + ib)D\{u + iv\} = aD\{u\} + ibD\{u\} + aD\{iv\} + ibD\{iv\}$$

By comparing, we must have $ibD\{u\} + ibD\{iv\} = bD\{iu\} - bD\{v\}$. If $D\{if\} = iD\{f\}$, then the above holds true. Also if the above holds true, then on letting $u = f, v = 0, b = 1$ we have $D\{if\} = iD\{f\}$. **(1 mark for showing $D(\alpha g) = \alpha D(g) \implies D(ig) = iD(g)$ or vice versa)**

Thus an $\mathbb{R}^2$-derivative D when extended to $\mathbb{C}$ is a $\mathbb{C}$-derivative iff $D\{if\} = iD\{f\}$. So,

$$iD\{f\} \equiv D\{(u, v)\}(0, 1) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{pmatrix}$$

$$D\{if\} = 1D\{-v + iu\} \equiv D\{(-v, u)\}(1, 0) = \begin{pmatrix} -\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial x} \end{pmatrix}$$

Thus, we need $u_x = v_y$ and $v_x = -u_y$ **(1 mark for $u - v$ relations)** for an $\mathbb{R}^2$-differentiable function to be a $\mathbb{C}$-differentiable function.