The Mathematics Club

Presents:

Be Real

Are you even continuous bro?

Do you keep on hearing about ‘Real Analysis’? :’)

What is **Calculus** used for?

In Real analysis, we deal with the **foundations of calculus**.
We ask pointless questions, like :

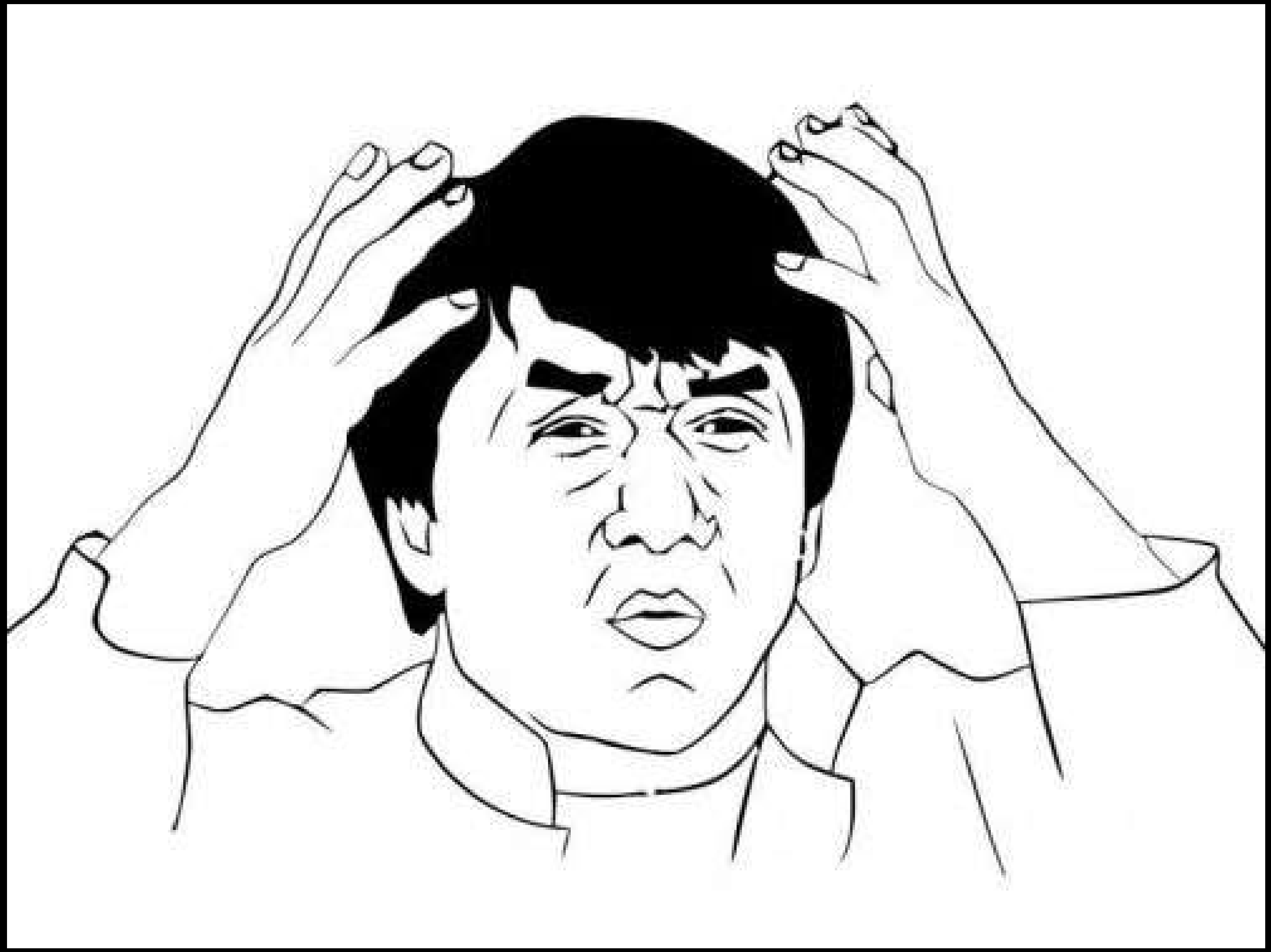
1. What is a real number?
2. Why do u need it for real analysis?

etc.

It wasn’t long before mathematicians realized they should not worry about what a number really is, but how to actually use it in practice. :)

What is Analysis?

Analysis is simply the study of **analytical functions**



But bro, Whats an analytical function??

Taylor series of a function : $$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

The function f is **real analytic at a** if there is some R > 0 so that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

when | x - a | < R

Lets go back to the basics : Limits, Continuity and Differentiability

Functional Limits :

Let $f : A \to \mathbb{R}$ and let c be a limit point of A. Then we say $\lim_{x \to c} f(x) = I$ if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $x \in A$ for which $0 < |x - c| < \delta$, we have

$$|f(x) - L| < \varepsilon.$$

In this case, we also say that $\lim_{x \to c} f(x)$ converges to L.

Continuity:

A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $x \in A$ where $|x - c| < \delta$, we have

$$|f(x) - f(c)| < \varepsilon.$$

In other words, f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

(provided c is a limit point of A).

Differentiability:

Suppose $f : A \to \mathbb{R}$. The derivative of f at $a \in A$ is defined to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If this limit exists, we say f is differentiable at a.

Max - Min

Definition: For a subset $M \subseteq \mathbb{R}$:
$b \in \mathbb{R}$ is called an upper bound for M if

$$\forall x \in M : \quad x \leq b.$$

$a \in \mathbb{R}$ is called a lower bound for M if

$$\forall x \in M : \quad x \geq a.$$

If b is an upper bound for M and $b \in M$, then b is called a maximal element of M. max(M)
If a is a lower bound for M and $a \in M$, then a is called a minimal element of M. min(M)

What if min(M) or max(M) is not defined?

Sup - Inf

Sup :

For a subset $M \subseteq \mathbb{R}$, a number $s \in \mathbb{R}$ is called the supremum of M if:

- $\forall x \in M: \quad x \leq s \quad$ (upper bound for M)
- $\forall \varepsilon > 0, \exists x \in M: \quad s - \varepsilon < x \quad$ ($s - \varepsilon$ is no upper bound for M).

Then write:

$$\sup M := s \quad \text{or} \quad \sup M := \infty \quad \text{if } M \text{ is not bounded from above.}$$

Inf :

For a subset $M \subseteq \mathbb{R}$, a number $l \in \mathbb{R}$ is called the <u>infimum</u> of M if:

- $\forall x \in M : \quad x \geq l \quad$ (lower bound for M)
- $\forall \varepsilon > 0, \exists x \in M : \quad l + \varepsilon > x \quad$ ($l + \varepsilon$ is no lower bound for M).

Then write:

$$\inf M := l \quad \text{or} \quad \inf M := -\infty \quad \text{if } M \text{ is not bounded from below.}$$

If $M = \emptyset$, then what is

$$\sup \emptyset := ?$$

and

$$\inf \emptyset := ?$$

Is there a notion of convergence just for Reals?

How about

Vectors?

A Sequence of Functions?

Observe that $|X-Y|$ for Real numbers gives a notion of distance

We do have distance defined b/w vectors.....

Let us try to reverse engineer stuff and find commonalities between the two...

Checks:

1. d(x,y) >=0 , 0 only when x = y
2. d(x,y) = d(y,x)
3. d(x,y) + d(y,z) >= d(x,z)

where d = distance

How about $|x_1 - x_2| + |y_1 - y_2|$

Now that we are convinced of metrics,
lets come back to our question of convergence

Can we redefine Convergence in terms of a Metric?

Now lets get to derivatives ...

Let's denote derivative of a function f at a point x as $\frac{d(f(x))}{dx}$

Note that

$$\frac{d(f(x)+g(x))}{dx} = \frac{d(f(x))}{dx} + \frac{d(g(x))}{dx}$$

$$\frac{d(c*f(x))}{dx} = c*\frac{d(f(x))}{dx}$$

Doesn’t this relate too much to vectors?

....

Matrices?

Indeed a whole class of objects called Linear Tranformations

Jacobians and MA1101, PH1010

Well, what info does Derivative give around a point?

$$f(x) = f(a) + \frac{d(f(a))}{dx}(x - a) + \ldots$$

$$y = mx + c?$$

CONVERGENCE!!

Let $f_1, f_2 ... f_n$ be a sequence of analytic functions: defined on the same domain

Let's say it converges to a function f(x)

If yes, what can we conclude about f(x)

Types for Convergence

Point wise Convergence:

$$||f_n(x) - f(x)|| < \epsilon \text{ for all } n \geq n_0$$

For all points in the domain

Types for Convergence

Uniform Convergence:

$$||f_n(x) - f(x)|| < \epsilon \text{ for all } n \geq n_0$$

For the entire domain

Wait, what’s the difference?

Types for Convergence

Locally Uniform Convergence:

$$||f_n(x) - f(x)|| < \epsilon \text{ for all } n \geq n_0$$

For all points in a compact closed subset of the domain

Uniform vs Locally Uniform Convergence

Collection of all compact closed subsets of the domain may not span the entire domain

Thus a Uniformly Convergent Sequence is also locally uniformly convergent

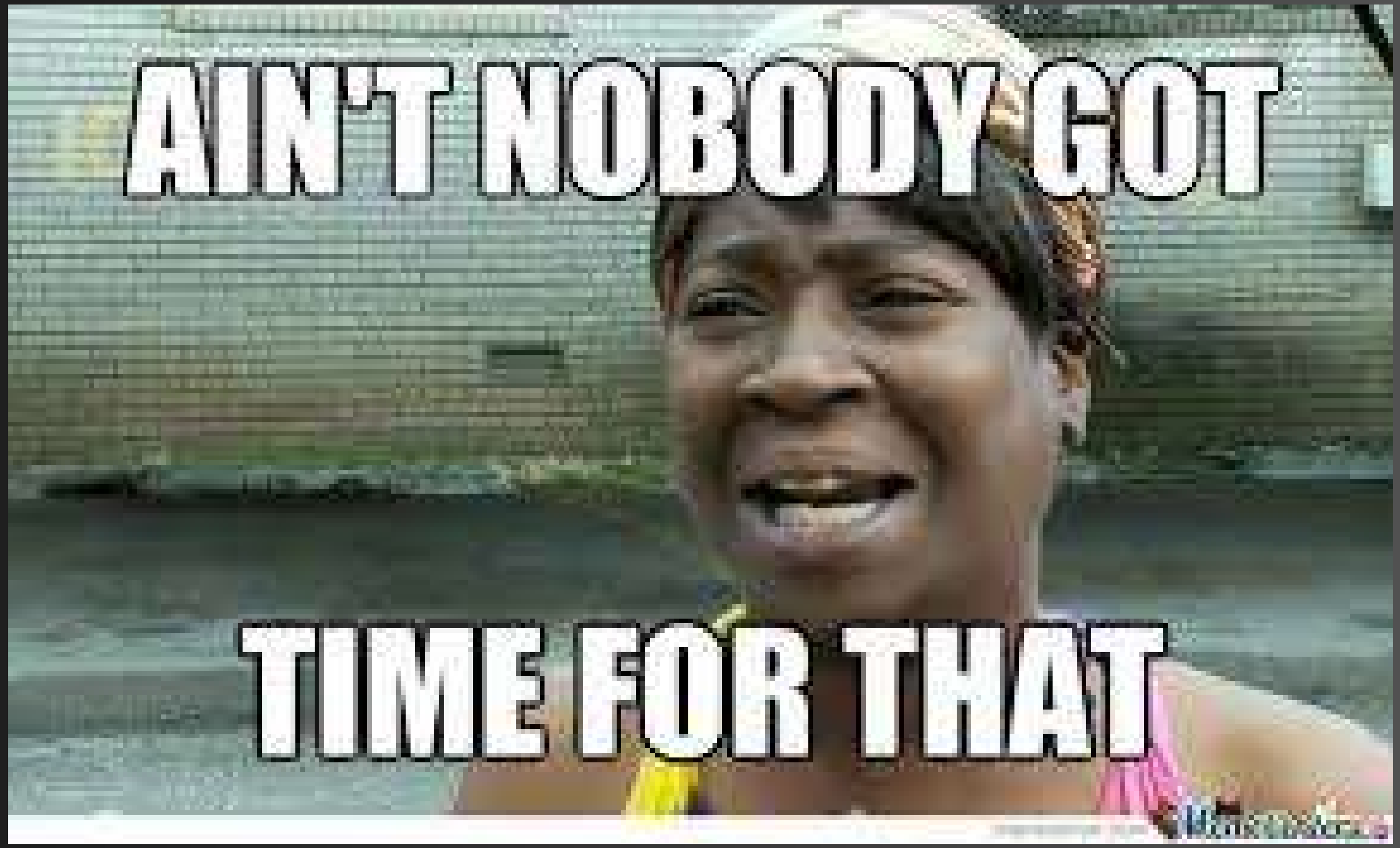
BUT

A locally uniformly convergent sequence may not be uniformly convergent

Uniformly convergent:
f(x) is analytic (i.e power series representation exists in the entire domain)

Locally uniformly convergent:
Power series representation exists in all the closed subsets of the domain but may not exist on the entire domain.
i.e Function may not be analytic.

Too much of painful solving with Epsilon (T.T)



Types for Convergence

Normal Convergence:

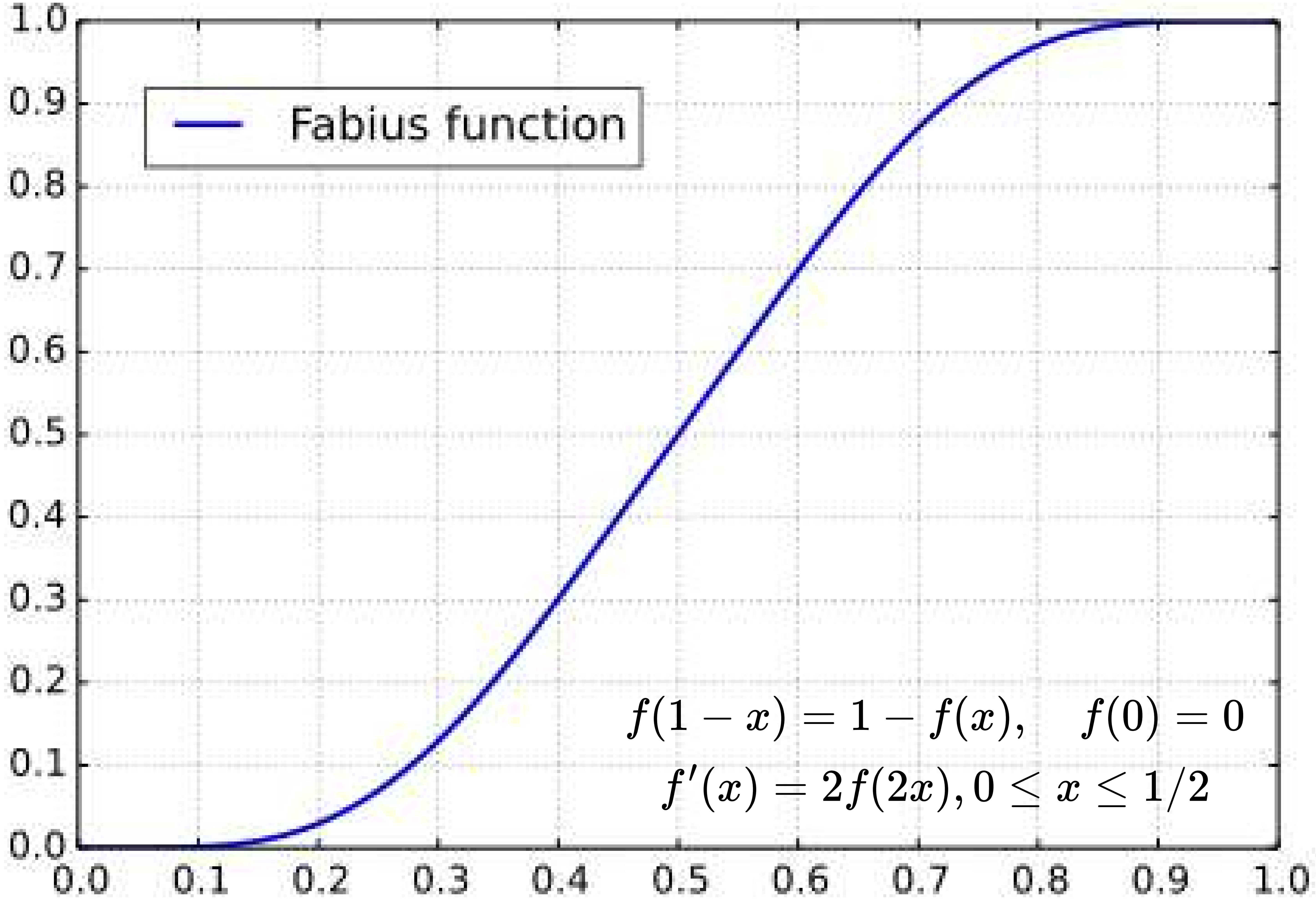
When the sequence of norms:

$$||f_1(x)||, ||f_2(x)||\dots||f_n(x)||$$

Converges for the entire domain

Is a sufficient (but not necessary) condition for uniform convergence (i.e a normally convergent sequence is uniformly convergent)

Challenge: List out the Properties of Analytic Function



$$f(1-x) = 1 - f(x), \quad f(0) = 0$$

$$f'(x) = 2f(2x), 0 \leq x \leq 1/2$$

$$f'(x) = 0 \iff x \in \{0, 1\}$$

$$f''(x) = 0 \iff x \in \{0, \frac{1}{2}, 1\}$$

$$f'''(x) = 0 \iff x \in \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$$

$$f^{(n)}(x) = 0 \iff x \in \{0, \frac{1}{2^{n-1}}, \frac{2}{2^{n-1}}, \dots, \frac{2^{n-1}-1}{2^{n-1}}, 1\}$$

$$\frac{k}{2^n}, k \in \{0, 1, 2, 3, \ldots, 2^n\} \quad \text{is dense in } [0, 1]$$

No open neigbourhood about rational where $f(x)$ has a non-polynomial Taylor Series

A smart hostel cat: “*A smooth function is analytic*”

You:

Theorems

$$f(a) = f(b) \implies f'(c) = 0 \quad \text{for some } c \in (a, b)$$

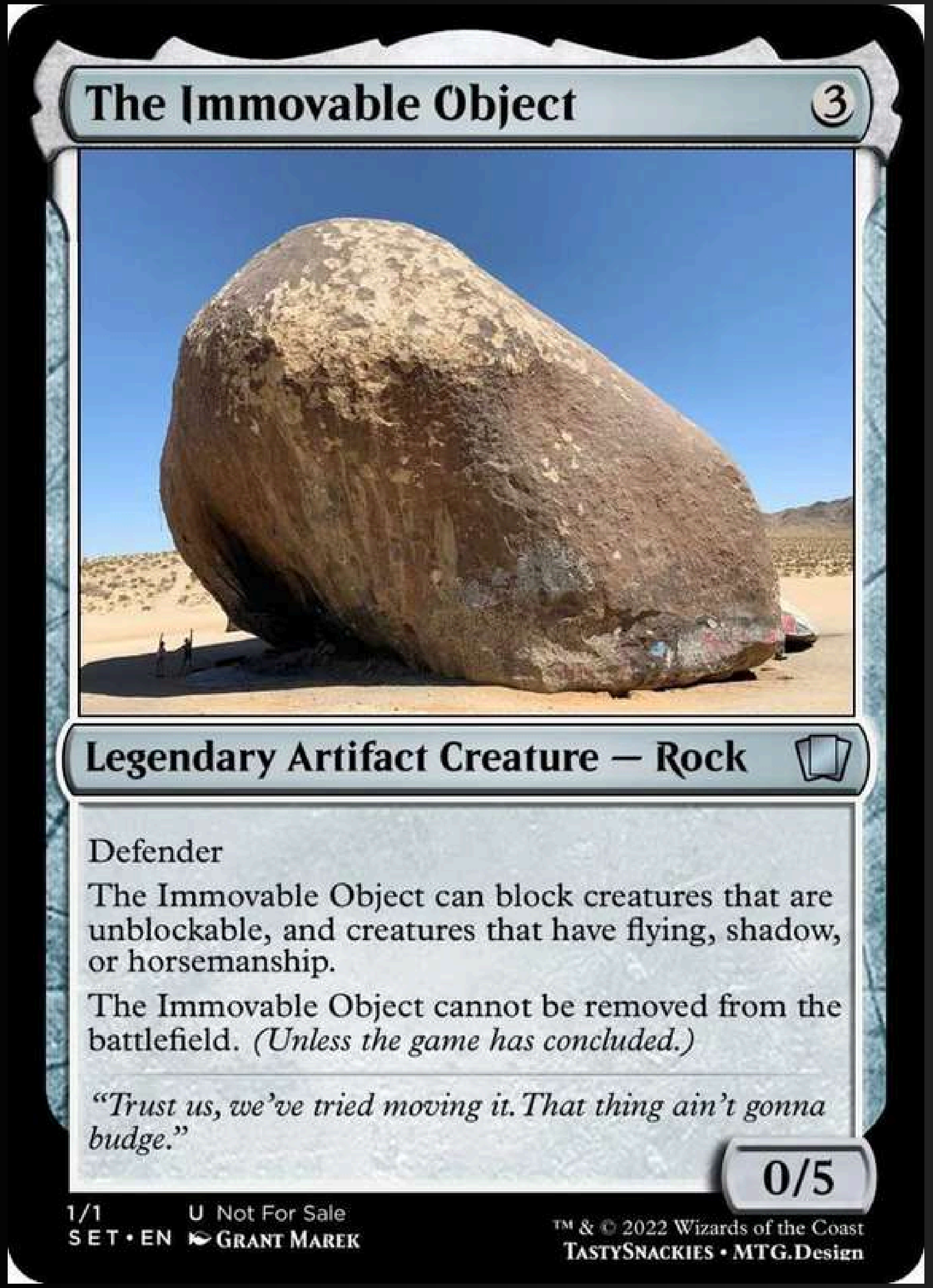
f is continuous in [a, b] and differentiable in (a, b)

$$\text{Continuity and differentiablilty in a finite region} \implies \text{Boundedness}$$

$$\text{Boundedness} \implies \text{Maximima/Minima/Constant}$$

At an extremum, left hand 'derivative'
and right hand 'derivative' are of opposite signs

$$\text{But the function is differentiable} \implies f' \text{ takes the value 0 there}$$


The Immovable Object
3
Legendary Artifact Creature — Rock
Defender
The Immovable Object can block creatures that are unblockable, and creatures that have flying, shadow, or horsemanship.
The Immovable Object cannot be removed from the battlefield. (Unless the game has concluded.)
"Trust us, we've tried moving it. That thing ain't gonna budge."
0/5
1/1 U Not For Sale
SET • EN Grant Marek
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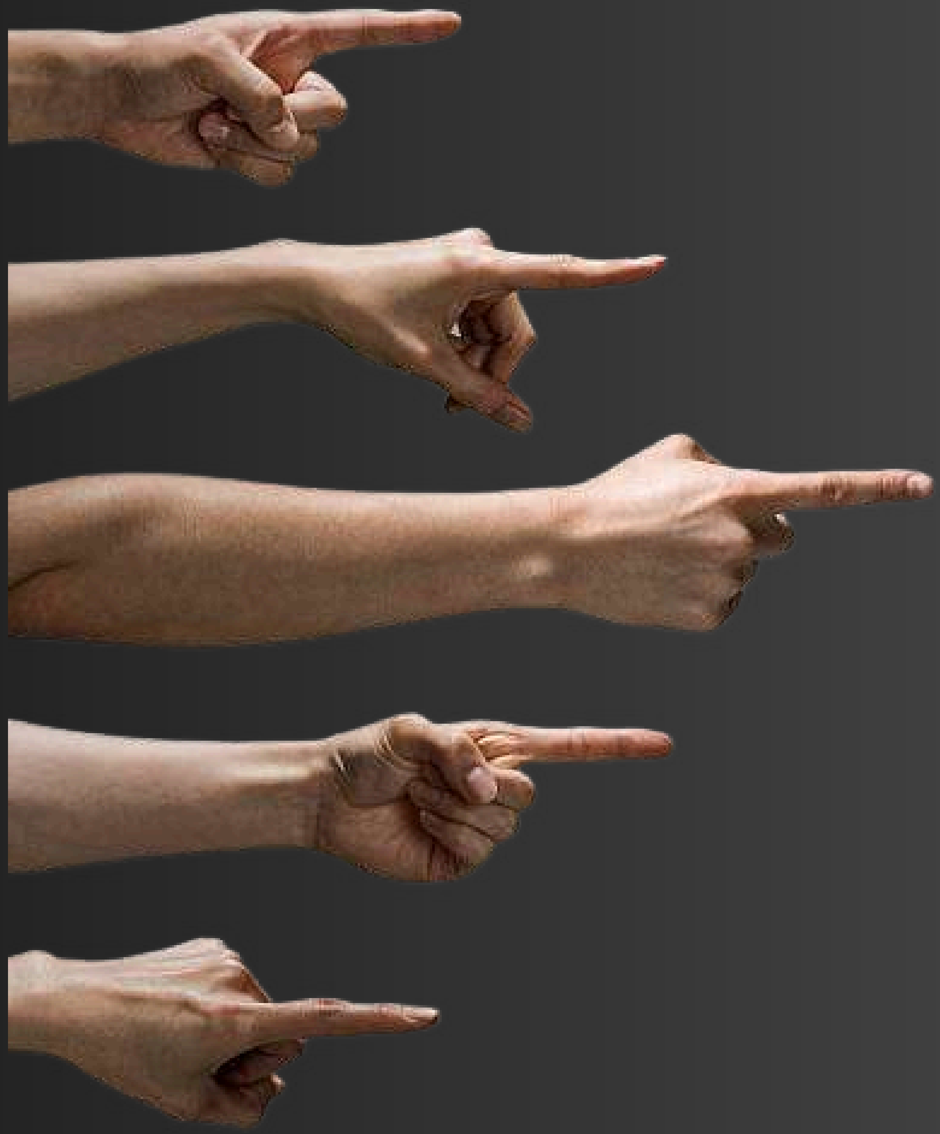

Theorems

$$T : X \to X, \quad ||T(x) - T(y)|| \leq q||x - y||, \quad q < 1$$

Contraction map (brings things closer)

$T(x) = x$ must be true for some x

(T has a fixed point)

Construct a sequence $\{x, T(x), T(T(x)), \ldots, T^n(x), \ldots\}$ and show that a limit exists to this sequence

Repeatedly apply triangle inequality to $||T^n(x) - T^m(x)||$ and show that the sequence is Cauchy

Any Questions?

THANK YOU