Advent Calendar 2024

Questions

Mathematics Club

IIT MADRAS

v1

§ Pentaweek 1 (01-05 Dec)

Day 1 | Let S be a non-empty finite set of positive integers satisfying:

$$\left(\frac{1}{i}+\frac{1}{j}\right)\times \mathrm{LCM}\,(i,j)\in S \quad \forall\; i,j\in S$$

Find the sum of the cardinalities of all possible S.

Day 2 | Pranjal has a challenge for Achintya. Let ε denote the non-terminating decimal expansion of a fraction (Take $\frac{1}{4}$ as an example, its ε is $0.24\overline{9}$). Pranjal asks him to write the ε of $\frac{1}{k}$, for any natural number $k>1$. Achintya is then required to find the number of digits in the non-repeating part of ε. (It is 2 ($0.\mathbf{24}\overline{9}$) in the case of $\frac{1}{4}$). Can you help him?
What would be the number of digits in the non-repeating part of ε for k $= 2^{1234}\times 3^{2345}\times 4^{3456}\times 5^{4567}$

Day 3 | Let $f:\mathbb{N}\to\mathbb{R}$ which satisfies the following property:
$f(1)+2^2f(2)+3^3f(3)+4^4f(4).....n^nf(n)=\left(\frac{n^{n+1}}{3}+\frac{n^n}{2}+\frac{n^{n-1}}{6}\right)f(n)$.

Find $\log_{10}\left(\frac{f(100)}{f(1)}\right)$.

Day 4 | Pranjal is at the point $(-1,0)$. He can take a step of magnitude r only towards the right with probability density function (PDF) $f_R(r)=Ce^{-r}$ (where C is an appropriate constant to ensure that the generated PDF is valid).
At what point is he expected to be at after 1729 steps? Enter the x-value. (Hint: What should C be?)

Day 5 | Find number of all combinations (a,b,c,p) of positive integers a,b,c and prime number p such that

$$2^ap^b=(p+2)^c+1$$

Assume $b\le 5$ and $b\le c$. You might want to have a look at P-adic valuation and Lifting the exponent.

§ Pentaweek 2 (05-10 Dec)

Day 1 | Evaluate the sum, $f(\mathrm{x}) = \sum_{k=0}^{\infty} \left\lfloor \frac{2^k+x}{2^{k+1}} \right\rfloor$]. where ($\lfloor x \rfloor$) denotes the Greatest integer Function. Let $f(50.2)= n$. When 100 coins are tossed, what is the probability that exactly n are heads? Round your answer to two decimal places.

Day 2 | Let

$$f(n) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{k} \times \binom{n-k}{\lfloor \frac{m-k}{2} \rfloor} \times 2^k$$

where $\lfloor x \rfloor$ denotes the Greatest integer Function and $\binom{n}{r}$ denotes the number of ways to choose r items from a set of n distinct items. Find the value of

$$\sum_{n=0}^{\infty} \frac{1}{f(n)}$$

Round your answer to two decimal places.

Day 3 | There is a certain series whose 1st term, s_0, is $\frac{5}{2}$, and the i'th term s_i, is given by $s_i = s_{i-1}^2 - 2$ for $i \geq 1$. Compute,

$$\prod_{i=0}^{\infty} \left(1 - \frac{1}{s_i}\right)$$

Day 4 | Consider the Euclidean space $X = \mathbb{R}^n$ equipped with the metric (parameterized with the parameter $p \neq 0$)

$$d_p(x, y) = \left(\frac{(x_1 - y_1)^p + (x_2 - y_2)^p + \ldots.. + (x_n - y_n)^p}{n} \right)^{\frac{1}{p}}$$

First, ensure that this is a valid metric for all $p \neq 0$. If it is not, answer -1. If it is, then assume $n = 2$. If

$$s = \sup_{y=0, x_1=1, x_2>0} \{p : d_p(x, y) \text{ is convex in variable } p\}$$

then find $-2s$. 'Convex in p' means fix $x = (1, x_2)$ and $y = (0, 0)$ and examine the behaviour of the metric as p changes.
(Hint: plug $p = 1$, $p = -1$ and $p \to 0$ and see if the form of the metric seems familiar.)

References:

- Metric spaces
- Convexity
- Infimum and supremum

Day 5 | Let $\psi(k)$ denote the smallest positive integer such that for every $n \geq \psi(k)$ $(n, k \in \mathbb{Z})$ there always exists an integer of the form p^4 $(p \in \mathbb{Z})$ in the range $(n, k^2 n]$. Find the value of

$$\left(\sum_{k=2}^{2024} \psi(k) \right) - 20$$

§ Pentaweek 3 (10-15 Dec)

Day 1 | Let x, y, z, w be positive real numbers satisfying

$$(x+z)(y+w) = xz + yw$$

Find the smallest possible value of

$$P = \frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x}$$

Day 2 | Let S be the set of ordered pairs (x, y) such that

$$0 < x \leq 1, \quad 0 < y \leq 1,$$

and

$$\left\lfloor \log_2 \left(\frac{1}{x}\right) \right\rfloor \quad \text{and} \quad \left\lfloor \log_5 \left(\frac{1}{y}\right) \right\rfloor$$

are both even. Given that the area of the graph of S is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

The notation $\lfloor z \rfloor$ denotes the greatest integer that is less than or equal to z.

Day 3 | For a positive integer N , let $f_N(x)$ be the function defined by

$$f_N(x) = \sum_{n=0}^{N} \frac{N + \frac{1}{2} - n}{(N+1)(2n+1)} \sin((2n+1)x).$$

Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.(Round your answer to two decimal placehello

Day 4 | Let $d : \mathbb{N} \to \mathbb{N}$ be defined as follows: If $n = 2^k a_k + 2^{k-1} a_{k-1} + \cdots + 2^0 a_0$, where $a_i \in \{0, 1\}$, then, $d(n) = \sum a_i$.

Let,

$$S = \sum_{k=1}^{2024} (-1)^{d(k)} k^3$$

Let, $P = S \mod 2024$, $P = uvwt$ (where u, v, w, t are the digits of the number P).

Two players each roll two standard dice, first player A, then player B. If player A rolls a sum of t, they win. If player B rolls a sum of w, they win. They take turns, back and forth, until someone wins. What is the probability that player A wins? (Round your answer to two decimal places.)

Day 5 | Consider ΔABC right-angled at A with $BC = 5$. Let BC be divided into 7 segments of equal length. Let the segment containing the midpoint of BC be l.
Find $\frac{\tan \alpha}{h}$ where α is the angle subtended by l at A and h is the altitude from A.

§ Pentaweek 4 (15-20 Dec)

Day 1 | Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1 \le i,j \le n}$ be the matrix with entries defined as:

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \ge 0, \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose B is an $n \times n$ matrix with entries 0 or 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Determine the number of matrices B satisfying the above condition for any set of random numbers.

Day 2 | PV, KK, KV, and PJ play a game of tag. The game begins with PV being 'it' (PV is tagged initially), and each time some other player is tagged (Self-tags are not allowed). What are the number of distinct ways PV is 'it' again after 6 tags? (Note : PV can be 'it' in the middle also)

Day 3 | I have two strings x and y. String x is formed by using the letters of the word Atreya while y is formed by using the letters of the word Shivanshu. Let $n(i)$ denote the sum of the number of possibilities for x and y, when both are of length i. Find the sum of all possible values of $\gcd(n(j), n(k))$, where j and k are relatively prime.

Day 4 | Given $0 \le x_1 \le x_2 \le \cdots \le x_n$,

$$\limsup_{n\to\infty} \left(\frac{\sum_{i=1}^{n} \sqrt{\frac{\sum_{k=i}^{n} x_k^2}{i}}}{\sqrt{n}\sum_{j=1}^{n} x_j} \right) = ?$$

Day 5 | touching S_1, S_2 and C_0. In general, let C_n be the circle touching S_1, S_2 and C_{n-1} . Suppose that h_n is the distance of the centre of C_n from the x-axis and d_n is the diameter of C_n . Compute

$$\frac{h_{100}}{d_{100}}$$

§ Pentaweek 5 (20-25 Dec)

Day 1 | Let $a_1 = 1$, $a_n = \sum_{k=1}^{n-1}(n-k)a_k$, $\forall\ n > 1$. Find a_{100}.
(Note: Some internet use (or a good calculator) might be required for the exact value.)

Day 2 | The roots of a 6^{th} degree polynomial (of the form $x + iy$) lie on the ellipse $\frac{x^2}{2} + \frac{y^2}{3} = 1$ such that they divide the ellipse into 6 regions of equal area. One of the roots is given to be $x = \sqrt{2}$. Find the product of roots.
(Note: Some internet use (or a good calculator) might be required for the exact value.)

Day 3 | Let A and E be opposite vertices of a regular octagon. A frog starts jumping from the vertex A. From any vertex except E of the octagon, it can jump to either of the two adjacent vertices. When it reaches the vertex E, the frog stops and stays there. Let e_n be the number of paths of exactly n jumps that start at A and end at E. Find

$$\lim_{n\to\infty} \frac{e_{2n+2}}{e_{2n}}$$

Day 4 |

$$f(n) = \begin{cases} n-2, & \text{for } n > 3000; \\ f(f(n+5)), & \text{for } n \leq 3000. \end{cases}$$

Find the value of $f(2024)$

Day 5 | Let, $ab = \pi y$ Find the sum of first 6 significant figures of y.

$$a = \lim_{n\to\infty} \sum_{k=2}^{n} \ln\left(\frac{2^k}{2^k - 1}\right) \prod_{p|k} \frac{p-1}{p},$$

where the product is taken over all prime divisors p of k.

$$b = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{2^k}{2^k - 1}\right) \sum_{d|k} \frac{f(d)}{d},$$

where the summation is taken over all divisors d of k,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a square-free positive integer with an even number of prime factors,} \\ -1 & \text{if } x \text{ is a square-free positive integer with an odd number of prime factors,} \\ 0 & \text{if } x \text{ has a squared prime factor.} \end{cases}$$

A square-free positive integer is a positive integer that is not divisible by any perfect square other than 1