Mathematics Club
Contingent Problem Set - 9

Challenge posed on: 18/10/2024 **Challenge conquered by: 28/10/2024**

1 Overview

- **Topics focused:**
  - Combinatorics
  - Inequalities
  - Number Theory
- **Challengers:**
  - Karthikeya
  - Saandeep
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Problems are not as scary as they look. You just need to find the correct key to open the lock.
- Please try to avoid Google/GPT etc., sources as much as you can. Instead, use your brain :)
- Happy solving :)

2 Problems

1. **Warm up** .
   (a) What is the reminder when 22! is divided by 23×11?
   (b) Find all possible values of n such that n and $\sqrt{n^2 + 2024n}$ are integers.
   (c) Which is larger: $2^{100!}$ or $(2^{100})!$?

2. **Scary long inequality!** Let $a, b, c \in \mathbb{R}^+$. Prove that the following inequality holds.

$$\frac{1}{3a} + \frac{1}{6b} + \frac{1}{9c} + \frac{3}{a+2b+3c} \geq \frac{1}{2a+2b} + \frac{1}{a+4b} + \frac{1}{a+6c} + \frac{1}{2a+3c} + \frac{1}{2b+6c} + \frac{1}{4b+3c}.$$

   When does the equality hold?

3. **Geoinequality.** Consider a triangle ΔABC and a straight line L which is not parallel to any of the three sides of the triangle. Let L cuts the sides BC, CA and AB at D, E and F respectively, which are different from the vertices. Then show that:

$$\frac{AF \cdot BD + FB \cdot DC}{AF \cdot DC + FB \cdot DC} + \frac{BD \cdot CE + DC \cdot EA}{BD \cdot EA + DC \cdot EA} + \frac{CE \cdot AF + EA \cdot FB}{CE \cdot FB + EA \cdot FB} \geq 3.$$

   When does the equality hold?

4. **I'm bored. Sorry :)** Let $x_i > 0$, where $i \in 1, \cdots, 2024$, be the zeros of the polynomial

$$x^{2024} + \sum_{k=1}^{2024} a_k x^{2024-k}.$$

   Then prove (or disprove) the following statements.

(a)
$$\frac{2}{a_{2024}} - \frac{3}{a_1} \geq \frac{25}{2a_{2024} - 3a_1},$$

(b)
$$\sqrt{506}\sum_{k=1}^{1000} x_k^{1.5} \geq 0.5(-a_1)^{1.5},$$

(c) The triplet
$$(x, y, z) = \left(\frac{x_1}{x_2 + x_3}, \frac{x_2}{x_1 + x_3}, \frac{x_3}{x_1 + x_2}\right)$$
satisfies $xy + yz + zx + 2xyz = 1$,

(d) Assume that there exist three roots such that the triplet can form an acute-angled triangle. Let α, β, and γ be the angles of this triangle. Then,
$$4\left(\cos^2\alpha\cos^2\beta + \cos^2\beta\cos^2\gamma + \cos^2\gamma\cos^2\alpha\right) \leq \cos^2\alpha + \cos^2\beta + \cos^2\gamma.$$
Hint: Use the identity $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$ if $A + B + C = \pi$ and use appropriate substitution

5. **Erdös-Ginzburg-Ziv theorem** Prove that from any $2n-1$ integers one can choose n whose sum is divisible by n.

6. **Mental Maths** Try to come up with combinatorial arguments and prove the following.

- $\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$
- $\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \cdot \binom{n}{k-i}$
- $\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}$
- $\left(\binom{n}{k}\right) = \sum_{m=0}^{n} \binom{n}{m} \cdot \left(\binom{m}{k-m}\right)$
- $\left(\binom{n}{k}\right) = \sum_{m=1}^{n} \left(\binom{m}{k-1}\right)$
- $k \cdot \left(\binom{n}{k}\right) = n \cdot \left(\binom{n+1}{k-1}\right)$

The term $\left(\binom{n}{k}\right)$ indicates the number of ways of forming a multi-set of size k from a collection of n elements. Try to count some quantity in two different ways.

7. **Menage Problem**

- Find the number of binary sequences of length $2n$ which has r 1s and $2n - r$ 0s. Such that no two ones are adjacent to each other, where the first and last places are also considered to be adjacent. Say this is $S(n, r)$
- There are n couples in a party, they want to sit around a circular table in $2n$ chairs(numbered). Under the following constraints.
  - The men and women should be alternate.
  - No husband should be adjacent to his wife.

  Show that the number of ways this can be done is $2 \cdot n! \cdot \sum_{k=0}^{n} (-1)^k \cdot (n-k)! \cdot S(n, k)$