Mathematics Club
Contingent Problem Set - 7

Challenge posed on: 26/07/2024 **Challenge conquered by: 02/08/2024**

1 Overview

- **Topics focused:**
  - Combinatorics
  - Number Theory
  - Inequalities
- **Challengers:**
  - Kirtan
  - Naveen
  - Rithika

2 Problems

1. **Complex tiling** A 49×49 square board is tiled using 800 1×3 rectangles. What are the possible positions of the unit square uncovered ? (A 1×3 rectangle can be placed either vertically or horizontally on the board).

2. **Easy ones** a) Let $x(n)$ denote the greatest proper divisor of natural number n. Find all $n \in \mathbf{N}$ such that $n + x(n)$ is a power of 10.

   b) Let m and n be two positive integers such that $m + k$ divides $n + k$ for all positive integers $k < m$. Prove that $m - k$ divides $n - k$ for all positive integers $k < m$.

3. **Unfair Inequality** Let a, b, c be positive real numbers such that a + b + c = 3. Prove that

$$[\sum_{cyc} \frac{a}{\sqrt[3]{4(b^6 + c^6) + 7bc}}] + \frac{1}{12} \sum_{cyc} \sqrt[3]{a} \geq \frac{7}{12}$$

   Here, $\sum_{cyc}$ means the cyclic sum, so each sum has three terms.

4. **Can Engineers Paint?** There is a street with n houses on the left side and n houses on the right side of a street. We have to paint the houses with m different colors in such a way that no two neighboring houses nor two face to face houses are of the same color. In how many ways can this coloring be done? (All the houses are identical and spaced in such a way that they perfectly face one house opposite to itself)

5. **Truncated power challenge** Find all numbers of the form 2^k such that, after removing the first digit of the number, the remaining number is still a power of 2.

6. **Is it really easy though?** Solve in integers:

$$4x^2 + 2xy + y^2 = \left(\frac{2x + y}{3} + 1\right)^3$$