Mathematics Club
Contingent Problem Set - 5

Challenge posed on: 12/07/2024 **Challenge conquered by: 19/07/2024**

1 Overview

- **Topics focused:**
  - Combinatorics
  - Linear Algebra
  - Calculus
  - Number Theory
- **Challengers:**
  - Aprajithan
  - Arya
  - Aravind

2 Problems

1. **Warm Up** (a) Consider 3 real numbers x,y,z such that $x + 2y + z = 6$.
   Find the minimum value of $x^2 + 2y^2 + z^2$.
   (b) Find the following indefinite integral:
   $$\int \frac{x^2 + n(n-1)}{(xsinx + ncosx)^2} dx$$
   where n is a natural number.

2. **A convergence test** Can there exist a convergent series $\sum a_n$ such that $\sum \frac{1}{n^2 a_n}$ is also convergent?

3. **The Cubic Polynomial** The polynomial $ax^3 + bx^2 + cx + d$ has integral coefficients a, b, c, d with ad odd and bc even. Prove that all the roots of this polynomial cannot be rational.

4. **Time for an inequality** If
   $$(x-1)y^2 + 4xy - 4y + 5x - 16 = 0$$
   $$xy^2 - 6xy + 10x - 28 = 0$$
   then
   $$2y(x-1) - x(y+1) \leq k$$
   Find the minimum value of k.

5. **Determinant ?!** Consider two matrices A and B with positive entries having sizes 3x2 and 2x3 respectively.Given the product matrix AB as $\begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{bmatrix}$ and BA is non singular.
   Find the determinant of the matrix BA.

6. **Catalan Frenzy !** (a) We define an increasing lattice path on a 2x2 grid as a path where each step taken is of unit length and is either upwards or to the right. Find the number of increasing lattice paths from (k,0) to (n+k,n+k) such that you never cross the $x = y$ line where $k \geq 0$ and $n \geq 1$ are integers.
   (b) Using the result from part (a), determine:

i. The number of rooted binary trees with $n \geq 2$ leaves such that each node has either both its left and right children or no child at all(i.e. It is a leaf).

ii. The number of raintrees with n nodes.
We define a raintree as follows:
A raintree with 1 node is just the root.
A raintree with n nodes consists of a root node and some smaller raintrees joined to the root node such that the total sum of the number of nodes in the joined raintrees is $n-1$ and that the order of their connection to the root node matters.

7. **Can you solve these ?** (a) Find all positive integer quadruples (x,y,z,w) satisfying $x^2+6y^2 = z^2$ and $6x^2 + y^2 = w^2$.

   (b) Find all solutions in positive integers to $x^8 + y^8 = z^6$.

8. **Done anything like this before, have you ?** We call a subset of a set neat if the arithmetic mean of the elements in the subset is an integer. Let a_n denote the number of neat subsets of the set of the first n even numbers. Prove that $a_n - n$ is always even.

9. **NumberFizz** The fizz of a number is defined as follows:

   The fizz of a single digit number is the number itself.

   The fizz of any other number is the same as the fizz of the sum of the digits of the number.

   We denote the fizz of a number n by $f(n)$.

   Consider the following function on x defined for $0 \leq x \leq 1$

   $f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n$.

   Find $\sum_{n=1}^{n=20} f_{p_n}(x)$ where p_n is the n^{th} prime number.