



Mathematics Club

Contingent Problem Set - 4



Challenge posed on: 05/07/2024

Challenge conquered by: 11/07/2024

1 Overview

- **Topics focused:**
 - Probability
 - Linear Algebra
 - Geometry
- **Challengers:**
 - Shriram
 - Nandha
 - Madhav

2 Problems

1. **Probably Geometry** Let ABC be a triangle and P be any point on BC . Let $\angle APB = \theta$, calculate the expected value $\mathbb{E}[\cos(\theta)]$.
2. **Determinant Dilemma** Solve these interesting determinants ($n \in \mathbb{N}$).
 - (a) Let $A_n \in \mathcal{M}_n(\mathbb{N})$ with $a_{ij} = \max(i, j)$. Calculate $\det(A_n)$.
 - (b) Let $A_n \in \mathcal{M}_n(\mathbb{N})$ with $a_{ij} = \sum_{k=1}^n k^{i+j}$. Calculate

$$\lim_{n \rightarrow \infty} \frac{(\det(A_n))^{1/n^2}}{n}$$

- (c) Let $A_n \in \mathcal{M}_n(\mathbb{N})$ with $a_{ij} =$ number of common divisors of i and j . Calculate $\det(A_n)$.
3. **May the ODDS be with you :)** During vacation, Shriram got bored. So, he decided to play with n dice, say, D_1, D_2, \dots, D_n , that is each die is numbered from 1 to n . He throws each of the dice twice (in order from 1 to n). Find the probability that the total sum obtained from the $2n$ throws is odd, given that the probability for the k^{th} die D_k to show an odd number is

$$\frac{\sqrt{\frac{2^{k+1} - 2}{2^{k+1} - 1}} + 1}{2}$$

4. **Escape the Matrix** Consider $A \in \mathcal{M}_{2020}(\mathbb{C})$ such that

$$(1) \begin{cases} A + A^\times = I_{2020}, \\ A \cdot A^\times = I_{2020}, \end{cases}$$

where A^\times is the adjugate matrix of A , i.e., the matrix whose elements are $a_{ij} = (-1)^{i+j} d_{ji}$, where d_{ji} is the determinant obtained from A , eliminating the line j and the column i . Find the maximum number of matrices verifying (1) such that any two of them are not similar.

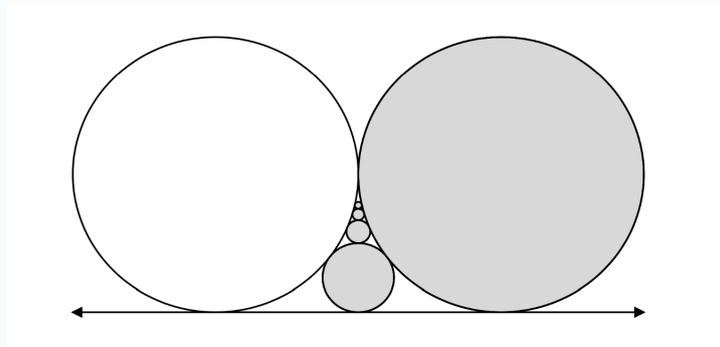
5. **What do you expect-double counting?** The Government has assigned 500 professors across the country to work with 100 different projects, with each project involving 50 professors. Prove that there exist two projects with at least 5 professors in common.

6. **Can you bound?** Consider the following determinant D_n

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}$$

Note that this is a $(n - 1) \times (n - 1)$ determinant. Is the set $\frac{D_n}{n!}$ bounded ?

7. **Circles all the way down** Two mutually tangent circles of radius 1 lie on a common tangent line as shown. The circle on the left is colored white and the circle on the right is colored gray. A third, smaller circle, is tangent to both of the larger circles and the line, and it is also colored gray. An infinite sequence of gray circles are inserted as follows: Each subsequent circle is tangent to the preceding circle, to the largest gray circle, and to the white circle. What is the total area bounded by the gray circles?



8. **Great Expectations** Let $p, n \in \mathbb{N}$. Let σ be a permutation of n objects chosen uniformly at random, and k be the number of **cycles** in σ . Find the expected value $\mathbb{E}[p^k]$ as a function of p and n .
9. **Enough with the Determinants** We consider two matrices $A, B \in \mathcal{M}_2(\mathbb{R})$, at least one of them is singular. Prove that $AB = BA = O_2$, if $A^2 + AB + B^2 = 2BA$.
10. **Lets end with some geogem** ABC is an inscribed triangle. CD is drawn parallel to AB and meets the tangent BD at D . AD cuts the circle at E . M is the midpoint of side BD . It is given that BCD is isosceles triangle with $BC = CD$. Prove that $\angle DEC + \angle BEM = 180$.

For the curious ones - BONUS

11. **Some more LinAl** Let $n \geq 1$ be an integer. Solve in $\mathcal{M}_2(\mathbb{Z})$ the equation $X^{2n+1} - X = I_2$
12. **Not again!** Let A, B and C be three points on a line (in the same order given) such that $AC = 26$ units and $AB = 5$ units. Let S be a circle passing through A and C such that AC is not a diameter of S . Suppose the tangents to the circle S at A and C meet at point R . Now RB meets the circle S at Q . It is given that $QB = 3$ and $QM = 9$, with M being midpoint of A and C . Now,
- (a) Prove that the point of intersection of angular bisector of AQC and AC is same regardless of the choice of S .
 - (b) Find the ratio of AQ and QC .