Mathematics Club
Contingent Problem Set - 2

Challenge posed on: 23/06/2024 **Challenge conquered by: 28/06/2024**

1 Overview

- **Topics focused:**
  - Polynomials
  - Complex Numbers
  - Induction
- **Challengers:** – Karthikeya
- The color coding is same as PS-1 but slightly easier than PS-1
- Note that 28/06/2024 is Friday

2 Problems

1. **Prime Polynomial** Let $a_1, a_2, \dots a_n$ be n distinct integers, where $n \in \mathbb{N}$. Let
$$P(x) = (x - a_1)^2(x - a_2)^2 \dots (x - a_n)^2 + 1$$
Prove that $P(x)$ is irreducible over $\mathbb{Z}$.

2. **Prime Polynomial 2.0 :)** Prove that the polynomial $P(x)$ is irreducible over $\mathbb{Z}$ $\forall n \geq 4$
$$P(x) = x^n + x^3 + x^2 + x + 5$$

3. **Pretty Pentagon!!** Let $z_1, z_2, \dots, z_5$ be 5 complex numbers on a circle such that $\sum_j z_j = \sum_j z_j^2 = 0$. Prove that $z_1, z_2, \dots z_5$ are the vertices of a regular pentagon.

4. **This is nameless and shameless** Let $P(x) = x^2 + ax + b$ be a real quadratic polynomial such that $a < 2$. Suppose the equation $P(P(x)) = 0$ has four distinct real roots and the sum of some two of these is less than -1. Prove that $P(x + y) \geq P(x) + P(y)$ for all reals x, y.

5. **I am not what I am** Do there exist 4 polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients such that the sum of any three of them always has a real root, but the sum of any two of them has no real roots?

6. **Enjoy the easy one** Let x_0 be a nonzero real root of the polynomial $ax^2 + bx + c$, where a, b, c are integers and at least one of b, c is nonzero. Prove that
$$|x_0| \geq \frac{1}{|a| + |b| + |c| - 1}$$

7. **Maximal Criminal** You are given an $m \times n$ grid of real numbers. Call an individual row or column a line. In a move, you are allowed to select any line, and flip the signs of each number in this line. Prove that you can make a finite sequence of moves, such that the sum of entries in any line is nonnegative at the end.

8. **Packing Problem** You're about to go on a vacation and have several items you need to pack. Each item weighs between 0 and 1 units, and the total weight of all items is n units, where n is a given positive integer. Each item must be placed in a bag, and the weight limit of any bag is 1 unit. What is the maximum number of bags you may need, in terms of n?