Mathematics Club
Contingent Problem Set - 1

Challenge posed on: 16/06/2024   **Challenge conquered by: 23/06/2024**

1 Overview

- **Topics focused:**
    - Double Counting
    - A bit of Graph Theory
    - Inequalities
    - Invariance

- **Challengers:**
    - Karthikeya

- The color of the problem roughly indicates the difficulty of the problem.

    - Cyan == Easy to Moderate
    - Blue == Moderate to Hard (few people might feel easy as well)
    - Red == kinda Very Hard

2 Problems

1. **Blackboard Odyssey: The Last Number**   The 2024 numbers $\{1, 2, 3, \ldots, 2024\}$ are written on a blackboard. We are allowed the following move: choose any two numbers on the blackboard, say a and b, replace them by the number $16ab - 176(a + b) + 1947$.

    After performing 2023 such operations, a single number α remains on the board. Determine all the possible values of α.

2. **Expression in Disguise**   Determine the smallest number k such that the inequality
    $$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \le M(a^2 + b^2 + c^2)^2$$
    holds for all real numbers a, b, c.

3. **Equality when not Equal** a, b, c are real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Prove that
    $$a^2b^2c^2 \le \frac{1}{54}$$
    When does the equality hold?

4. **Well-known in Disguise**   Let a, b, c be side-lengths of a triangle. Prove that
    $$a^2 \left(\frac{b}{c} - 1\right) + b^2 \left(\frac{c}{a} - 1\right) + c^2 \left(\frac{a}{b} - 1\right) \ge 0$$

5. **The Monochrome Challenge**   Six points, no three of which are collinear are placed in a plane. Every pair of the points are joined by a segment. Each segment is colored with either red or blue. Show that at least two monochromatic triangles are formed with vertices in the 6 points.

6. **Melodious Mystery**   Eight singers are attending a concert. They participate in m shows such that four of the singers participate in each of the shows. Any pair of singers participate in the same number of shows together. Find the least number of shows.

7. **The Handshake Riddle (Sorry DCs!!)**   At a party, some people shake hands. The following are known:

   - Each person shakes hands with exactly 20 persons
   - For each pair of persons who shake hands with each other, there is exactly one other person who shakes hands with both of them
   - For each pair of persons who do not shake hands with each other, there are exactly six other persons who shake hands with both of them

   Determine the number of people in the party.

8. **The Polychrome Challenge**   There are 18 points in plane, no three being collinear. Connect each pair of points with a segment, and color each segment red or blue. It is known that some point A has an odd number of red segments, and the other 17 points have different numbers of red segments. Find the number of red triangles in the graph and the number of triangles with exactly two red edges.

9. **Titu-Magic**   Let a, b, c be nonnegative real numbers such that

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} \geq 1$$

   Prove that

$$\frac{1}{5a+1} + \frac{1}{5b+1} + \frac{1}{5c+1} \geq \frac{1}{2}$$

10. **Enjoy the Easy one**   Do there exist a sequence of 1729 consecutive positive integers containing exactly 42 primes?